Creative approaches to practising mathematical procedures

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Introduction

Why are school mathematics lessons so boring? Perhaps that question is a little unfair. Not everyone finds mathematics lessons boring all the time; some students love their mathematics lessons. But many students do not seem to derive much satisfaction from their study of mathematics. Adults often complain that it was their least favourite subject at school. Why should this be? One reason often given is the experience of completing page after page of repetitive exercises. In England, the books that students write in in their mathematics lessons are typically called 'exercise books', suggesting that exercises are the dominant written activity taking place in the classroom. And traditional mathematics exercises can be quite dull.

Why do schools and teachers often subject students to seemingly endless practice of prescribed procedures, rather than focusing lessons on inquiry, investigation and problem solving? The reason seems to be the need for developing mathematical fluency. Fluency in important mathematical procedures is essential if students are going to be in a position to move onto harder mathematics and tackle more complicated problems. Unless students have a well-stocked toolbox of essential techniques, they will be poorly equipped to engage with novel problems and scenarios. So procedural fluency is essential if students are to make progress with their mathematics.

People sometimes claim that in mathematics we don't need exercises, because provided that students are exposed to a rich enough diet of problem-solving tasks, over time they will encounter all of the important procedures that they need incidentally. This is a bit like saying that a musician does not need to practice scales (or work on etudes) because they can get all the necessary practice by just playing through a wide range of different pieces of music. I think that that doesn't generally work in music, because nowhere do you get the *intensive* focus on any one particular skill that we know is needed for the efficient development of expertise. Any particular piece of music will usually present a range of technical challenges. The musician might fumble their way through each one, without sounding too bad, but not really get any better at any of them. Consequently, over a period of time, their competence stagnates. To improve at any particular technique requires a clear focus on that one specific technique – otherwise our working memory is swamped by all the other things that we have to think about. This argument has traditionally led to the dominance of 'exercises' in mathematics, but if these are unimaginatively created then they are not satisfying to work through. Consequently, the brain may switch off, and very little is learned. At the same time, negative dispositions towards mathematics may develop, which make students reluctant to pursue the subject beyond the compulsory phase.

Mathematical etudes

The question that interests me is: Are traditional exercises the only effective way to develop fluency with important mathematical procedures? I have recently explored alternatives to exercises, which I call '*mathematical etudes*', by analogy with musical etudes. Etudes were originally pieces of music written principally to develop a particular specified technical skill, such as playing long runs of very quick notes on the piano. However, these musical etudes were often beautiful compositions in their own right. Chopin's etudes, for example, are marvellous pieces of music, which many people today listen to for pleasure without even knowing that they are 'etudes'. It could even be that the constraint of focusing on a particular technical challenge is partly responsible for the creativity of the music.

Thinking about musical etudes led me to ask: Could we have something similar in mathematics? And, if so, what would a *mathematical etude* look like? At www.mathematicaletudes.com/ I have begun to design and collect tasks which might warrant the name of 'mathematical etude' (Foster, 2013, 2014, 2017). Mathematical etudes seek to provide the necessary concentrated practice, but in a richer, more stimulating context. Here is an example, suitable for students aged about 11-14 years. Suppose that you

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want students to practise solving equations in which the unknown quantity appears on both sides. You could give them traditional exercises involving numerous questions like $3x – 5 = 2x + 7$. Alternatively, here is an etude that I think achieves the same thing – and more (Figure 1, [Foster, 2015]).

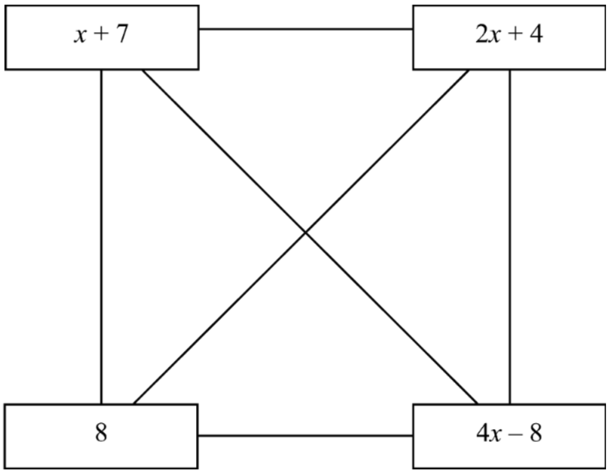

Figure 1. A four-expression polygon (Foster, 2015)

Figure 1 shows an ‘expression polygon’ (Foster, 2015) containing four algebraic expressions, with every expression connected to every other expression by six lines. Each line joining two expressions creates an equation. So, for example, the top line joining $x + 7$ and $2x + 4$ corresponds to the equation $x + 7 = 2x + 4$. The first task for students is to solve these six equations and write each solution next to the corresponding line. This is more than a set of six exercises, because of what comes next. There is a pattern to the solutions obtained: 1, 2, 3, 4, 5, 6 and this pattern is intended to be provocative. When students find the six solutions, they sometimes say, “That’s cool!” or “How did you do that?” Then the main task is for students to make up an expression polygon of their own that has a ‘nice’ set of solutions. (It is up to them to decide what counts as ‘nice’.)

Doing this is harder than it may look, and entails a lot of trial and error, with students picking expressions, solving the resulting equations, and then modifying their expressions, which generates a great deal of essential practice. Over time, this tends to develop into ‘intelligent’ trial and error, in which students unpick how the process is working so as to achieve the solutions that they are aiming for. Reverse-engineering the equation-solving process supports the development of their fluency. Sometimes they will begin with the four given expressions and modify them. For example, students aiming for 2, 4, 6, 8, 10, 12 may try doubling all of the expressions, only to find that this leads to the original set of solutions 1, 2, 3, 4, 5, 6. This can be a good discussion point. Alternatively, students aim for a completely different set of solutions, such as the first six prime numbers. Actually, even just aiming for integer answers can be enough of a challenge at the start.

The task is naturally differentiable by, for example, varying the number of expressions in the polygon; a triangle of three expressions can be a good place to start. At the opposite extreme, those comfortable with the equation solving may pose quite challenging questions, such as: What sets of 6 numbers is it possible to generate from a 4-expression polygon? Could you generate the numbers 2, 2, 2, 2, 2, 3, for instance? Some students working on this etude will mainly be focused on getting comfortable with the solving of the equations. However, as fluency develops, there are much wider problems available for students to think about, and increasingly students’ attention will shift from getting the procedure right to exploring how it works and investigating the structure of the expression polygons. In this way, the task is intended to embed useful repetition of a routine procedure into a self-differentiating, rich mathematical problem.

It is important to stress that etudes are intended to be used ***after*** the teacher has taught the necessary procedure. It is not expected that students will ‘discover’ the procedure by working on the etude. It is also worth noting that I do not envisage etudes as an 'extra' activity, for teachers to add in to their crowded curricula. There is always very little slack time in schools, and asking teachers to find lesson space for extra tasks is hard. An etude is intended to ***replace the exercises***, not be additional to them. The teacher teaches the topic as usual, but at the point where they would say "Now turn to page 52 and do questions 1 to 20" they instead say "Now do this etude". This means that the etude needs to be at least as effective as the exercises that it replaces, and it is an empirical question whether this is the case.

Empirical work

In a recent series of quasi-experimental studies (Foster, 2018), I trialled etudes against traditional exercises in two subject areas:

- an algebraic topic: solving linear equations in which the unknown quantity appears on both sides of the equation (using the etude described above), and
- a geometric topic: performing an enlargement of a given shape on a squared grid about a given centre of enlargement with a given scale factor.

Pre- and post- tests focused on performance of the procedure only. The intention was to see whether the etudes were as effective as traditional exercises at improving students' fluency with the particular procedure. It seems likely to me that etudes have many other, harder-to-measure benefits, such as developing conceptual understanding of the mathematical structure and developing students' problem-solving abilities. I also hope that etudes are more satisfying for students, and consequently more engaging. However, these claims are much harder to investigate, and so, for this initial series of studies, I focused purely on how good the etudes were at developing procedural fluency. This meant that the pre- and post- tests were strongly aligned to the traditional exercises that were given to the control group, meaning that any bias should be towards the control group rather than the etudes intervention group.

Bayesian modelling provided evidence across the studies of no difference between the effectiveness of the etudes and the traditional exercises in terms of developing procedural fluency. This means that even if all the teacher cares about is students' development of procedural fluency – and most teachers care about much more than this – they might as well use etudes. These studies suggest that we do not have to pay a price in development of procedural fluency for using richer, more interesting tasks, provided that they are designed in the way I have outlined above. Of course, these conclusions are based on early findings across only two topic areas. In my current research I am looking at a much wider set of etudes, and exploring how students engage with these tasks compared to when they work on traditional exercises.

Conclusion

In this article I have contrasted etudes with traditional exercises for the development of students' mathematical fluency. However, I do not claim that etudes are a completely novel idea. I think you could label some of the tasks available on a website such as https://nrich.maths.org/ as etudes if you wanted to. What I am trying to do in this work is to explore the potential of etudes as alternatives to traditional exercises. The empirical findings suggest, at least with those etudes trailed so far, that they are no worse than exercises in narrow terms of development of procedural fluency, and I think it is very plausible that etudes have many other important advantages.

Etudes have so far been very favourably received by teachers, whose main concern has been where they can find such tasks when they need them. I am building a collection at www.mathematicaletudes.com/, where you can find out more about the ***Mathematical Etudes Project***. If you prefer to listen than to read, then I was recently interviewed about this work on the *Mr Barton Maths* podcast: www.mrbartonmaths.com/blog/colin-foster-mathematical-etudes-confidence-and-questioning/.

References

Foster, C. (2013). Mathematical études: Embedding opportunities for developing procedural fluency within rich mathematical contexts. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 765–774. https://doi.org/10.1080/0020739X.2013.770089

Foster, C. (2014). Mathematical fluency *without* drill and practice. *Mathematics Teaching*, *240*, 5–7.

Foster, C. (2015). Expression polygons. *Mathematics Teacher*, *109*(1), 62–65.

Foster, C. (2017). Mathematical etudes. NRICH article available at: https://nrich.maths.org/13206

Foster, C. (2018). Developing mathematical fluency: Comparing exercises and rich tasks. *Educational Studies in Mathematics*, *97*(2), 121–141. https://doi.org/10.1007/s10649-017-9788-x