The devil in details: Mathematics teaching and learning as managing inter-discursive gaps

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The human capacity for communicating with one another may well be this one thing that makes us what we are. This said, our ability to understand each other is only too prone failure. Nowhere is the danger of communicational gaps greater than in mathematics classroom. In this WikiLetter I promote the thesis about the importance of our ability to notice such gaps.

The most dangerous feature of communicational gaps is that they tend to go unnoticed. To put it in the words of George Bertrand Shaw, "The single biggest problem in communication is the illusion that it has taken place". On the other hand, whether invisible or not, some gaps may be indispensable for learning. Indeed, the most substantial development in mathematical thinking takes place when the student reaches across an inevitable communicational fault line to a new territory, from which the things she thought she knew appear quite different than they did so far. Sensitizing the learner and the teacher to this latter type of pitfall is a critical step in turning the gap from an obstacle into an opportunity for learning. In what follows, after illustrating this thesis with an example, I will also claim that the communicational gaps, if noticed, may disclose the deeply hidden inner workings of mathematical thinking.

Gap-noticing as a necessary basis for effective teaching

In the classrooms, the danger of the *illusion* of communication is at its worst when nothing seems unusual and a communicational glitch, although quite real, does not manifest itself in a palpable way. Consider, for instance, the following exchange between a teacher and her student.

Multiplying by Fraction

	Speaker	**What is said**	**What is done**
1.	Teacher:	So, what is?	writes $^1/_3 \cdot 12$
2.	Student:	......	
3.	Teacher:	Try again, one third times twelve	
4.	Student:	I think.... Don't know...	
5.	Teacher:	Once again, one third of twelve	
6.	Student:	Ahm..... It's four	
7.	Teacher:	Great. See, when you think about it, you know how to do it!	

What happened here may appear so familiar that the claim about the student's initial difficulty as due to communicational issues is likely to be met with scepticism. Indeed, there is

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nothing surprising in the fact that the child, who is evidently quite new to the topic of fractions, has difficulty multiplying a fraction by a whole number. It is also understandable that after the teacher's additional probing (see turns [3] and [5]) and with some deeper thought on the part of the student, the proper answer is finally produced ([6]). Nor does anything seem questionable when the teacher summarizes saying that a bit of effort was all the boy needed to succeed ([7]).

Notice, however, that implicit in this last claim is the teacher's belief in the learner's acquaintance with the necessary procedure. According to this assumption, the child's momentary failure is a mere result of his as-yet insufficient proficiency in the required operations. As unproblematic as this simple account seems to be, at a closer look it leaves an important question unanswered. Yes, the child did seem to make an effort. Yet, although he clearly tried hard already the first time round, he was able to produce an answer only after the teacher's third inquiry. What was it about this third question ([5]) that brought the sudden insight? How was this query different from the previous ones ([1], [3])? Some scrutiny of the three instances may suffice to realize that each of the three utterances referred to the required operation in its own way:

- in [1], the operation was signalled it was the written expression '$^1/_3 \cdot 12$'
- in [3] it was done orally, with the expression 'one third times twelve'
- in [5], the teacher helped herself with the expression 'one third of twelve'.

Note the common feature of the two first cases: both the symbol '$\cdot$' and of the word 'times' are typical of the formal talk on numbers. Kin contrast, the expression 'one third *of* twelve" that appears in the third query comes from everyday talk, and is usually well known even to people with no experience with formal mathematics. In our example, therefore, the first two utterances directed the child to as-yet unfamiliar numerical operation, whereas the third one brought to his mind the everyday action of identifying a familiar part of a whole.

This latter interpretation implies a communicational gap between the teacher and the child, one that remained invisible to the participants: the three expressions that constituted "the same question" in the eyes of the teacher, were received by the student as defining two different tasks. The difference between this account and the one offered implicitly by the teacher, although quite subtle, is highly consequential. The teacher who views skilfulness as the child's only problem would be likely to put all her energy into fostering the child's procedural proficiency. In contrast, the realization that the boy might had thought about her questions in ways different from her own would have turned her attention to the conceptual side of the story. Had she arrived, in consequence, at an interpretation similar to the one offered above, she would have likely decided to focus on helping the learner to see connections between his everyday talk on fair sharing and the mathematical discourse on fractions.

The practical impact of gap-noticing is thus quite weighty, and as such, it may suffice to justify the claim about the importance of minding communicational gaps. I would not end this note, however, without presenting, in addition, some theoretical benefits of our constant communicational vigilance. In what follows, I argue that noticing hidden differences of interpretation such as the one exemplified above may lead to no less than novel insights about the mechanisms of learning mathematics.

Communicational gaps as windows to the mechanisms of learning mathematics

Let me begin these theoretical musings with conceptual clarifications. It may be useful to look at mathematical thinking as a *discourse* in which we engage whenever mulling over mathematical objects such as numbers, geometric figures, functions and the likes.[2] The word *discourse* is used here as referring to a special form of communication, characteristic of a particular community. In our present context, the community is that of mathematicians or of mathematics classroom. Note that in spite of the common reference to mathematics, these last two discourses are quite different: the one that is practiced in schools is not the same as the one to be found in mathematics departments and research journals.

Discourses are set apart by several characteristics. To begin with, each discourse has its own distinctive keywords, and only too often, the same words are used differently in different discourses. Another discourse-defining discursive feature is a set of its special recurrent ways of acting, known as *routines*. Calculating, differentiating, proving, and constructing geometric figures with ruler and compass are all good examples of well-defined mathematical routines. Note that some of these routines are algorithmic, whereas other ones escape deterministic descriptions.

Far from being just an optional way of acting (and a rather boring one, some may say), routine is what makes us able to act in the first place. To react to a prompt for an action in an immediate way, the best we can do is to turn to a *precedent* – to identify a familiar ways of acting that worked well in a similar situation in the past. In the example above, it was probably a successful search for a precedent that, eventually, made the student able to answer the teacher's question. Once we identify a suitable precedent, we decide with its help about, first, our *task*, that is, the goal of the action we are supposed to perform; and second, about a *procedure*, that is, the sequence of steps that suits that task. The relevant routine is the resulting *task-procedure pair*.

Endowed with this new conceptual tool, let me return to the former example and reformulate the results of my analysis: the three situations created by the teacher's questions [1], [3], and [5], although identical in the eyes of the teacher, led the student to search for precedents in different places. Following questions [1] and [3], he turned to past classroom situations in which whole numbers were multiplied by fractions with the help of a formal algorithm. Question [5], on the other hand, brought to his mind everyday situations in which a conversation was not about multiplying, but rather about sharing a certain amount of cookies fairly between three friends. The *tasks* envisioned by the child as a result of these differing choices of discourses and precedents were also different: In the first case, he saw it as his job to perform the symbolic manipulation he learned in school. In the second case, his task was to find out what would be the share of one person if twelve items were distributed evenly between three people.

From this example, we can arrive at an important insight about how routines develop. The first thing to notice is that the two tasks identified above, that of multiplying and that of finding a fair share, are strikingly different from one another. If this fact tends to escape us, it is probably because of our long acquaintance with the operation of multiplying by fractions and with its

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[2] To learn more about this approach, with its origins, conceptual infrastructure, applications and advantages, see, for instance, Sfard 2008, 2019.

numerous applications. Indeed, having mastered these routines to the point of automaticity, we may have long forgotten that we were probably capable of using words such as *half*, *quarter*, (*one*)-*third* or *three-quarters* well before we knew anything about the formal discourse on fractions. Similarly, we no longer remember that once upon a time, these basic fraction words did not function for us as names of numbers, but were rather labels for some special routines. At that time, "finding a third of a pizza" meant not much more than a physical action of cutting the pizza into three parts. Similarly, "giving each of the three friends *a third of* twelve cookies" meant the iterative action of handing one child after another a piece of pizza (usually while saying "one for you, and one for you…") until the set of twelve was exhausted. Indeed, we were well capable of performing this procedure long before the expression "$^1/_3 \cdot 12$" became meaningful for us or even before being able to perform the relevant operation. Thus, in the beginning, different rational numbers corresponded to different procedures used in execution of different tasks. It took time until the diverse tasks consolidated into one, and the dissimilar procedures became alternative branches of a single computational algorithm.

Such *bonding*[3] of several hitherto separate routines and turning them into a single one constitutes one of the central mechanisms in the development of discourses. In the present case, many other routines have yet to be bonded with the formal operation "$^1/_3 \cdot 12$" before the full-fledged, widely branching routine for multiplying rational numbers emerges. In the eyes of the beginner, many of these additional routines may initially appear as totally unrelated to the school discourse on fractions. The process of gradual bonding will lead to successive extensions in the applications of the resulting super-routine known as multiplication of rational numbers. These developments will greatly increase the usefulness of the multiplication routine, and with it, they will considerable enhanced the applicability of the whole discourse of rational numbers.

This one case, as restricted as it may appear, is nevertheless a good illustration of the eye-opening impact of gap-noticing. It highlights the importance of active, deliberate searching for mismatches between word uses and routines of beginners and the experienced participant of mathematical discourses.

References

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[3] This type of bonding, one that happens between different procedures, is sometimes qualified with the adjectives *horizontal* or *external* so as to be distinguished from the bonding that occurs inside the procedure, and is thus known as *vertical* or *inner* (see Lavie et al., 2019).