Learning to Teach

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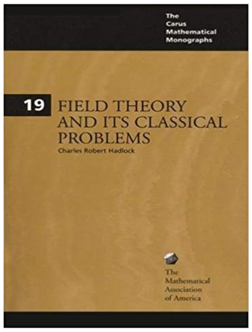
Learning how to teach is a process that continues, or should continue, throughout one's career. While there have been many moments of enlightenment throughout my career, I would like to take this opportunity to focus on just four.

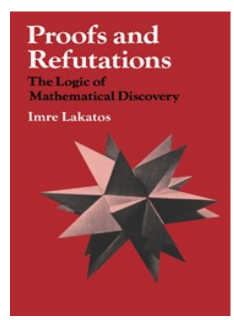


The first came early. I agreed to teach independent study to a mathematics major who need a course in Abstract Algebra but could not fit it into her schedule. I had recently read Charles Hadlock's *Field Theory and Its Classical Problems*. I loved the way it took the three classic problems for straightedge and compass—doubling the cube, trisecting an angle, and squaring the circle—and used them to motivate an entrance into ring theory. There are 24 sections and only a few problems for each. Her weekly assignment was to read several sections and attempt the problems.

At the start of the semester I dutifully worked out the solutions before we met, but as the press of other demands increased, I stopped being so well-prepared. Often the problems that she could not solve also caused me considerable difficulty.

At the end of the semester I apologized that so often I had not been prepared to explain the solutions. She told me that no, that was the most important part of the course, seeing how I thought through a problem I did not know how to solve. That was my first realization that teaching is less about laying out a clear explanation of the mathematics and more about helping students learn how to wrestle with challenging problems.

The second came after teaching a very discouraging first course in real analysis. After slogging through the definitions and theorems that my students vainly attempted to memorize and apply to artificial situations, we arrived at the chapter on Fourier series. I was excited to display how almost everything we had studied that semester came together to provide insight into these strangely behaved infinite sums. My class was unimpressed by my use of results they only vaguely remembered. As I thought about this failure, I realized that I had taught this course exactly backwards. Historically, questions about Fourier series had come first, and most of the structure of 19th century analysis, both definitions and theorems, had arisen from the effort to answer these questions. At about the same time, I read Imre Lakatos's *Proof and Refutations*. The first Appendix discusses Cauchy's "proof" that every infinite series of continuous functions is itself continuous. I saw in this an opportunity to illustrate for my students the difficulties that mathematician's encounter in exploring new mathematics, at the same time building their appreciation for the



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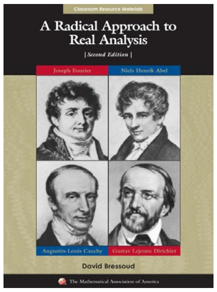
role of precise definitions. I made Cauchy's "mistake"[2] the centerpiece of the re-envisioned course.



The first time I taught real analysis this way, one of my students objected that if Cauchy could be wrong, how do we know that anything we are taught in mathematics class is correct? I have come to realize that this is a core question I want all of my students to ask, not as a complaint, but as the start of their own exploration of what it means to learn mathematics. That course laid the foundation for my own real analysis textbook, *A Radical Approach to Real Analysis.*

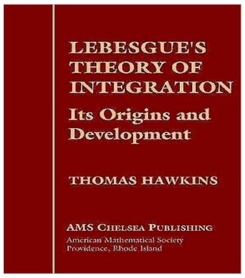
I was fortunate during my last year at Penn State that David Smith, one of the authors of *Project CALC* (Calculus as a Laboratory Course) was also there. I used his materials for the entire year for an honor's section of single variable calculus. The structure of the course involved three hours per week, Monday-Wednesday-Friday, in a classroom and two hours per week, Tuesday-Thursday, in a computer lab using MathCAD.[3] The course ran on a regular rhythm: introduction of a new concept in class, the opportunity to explore it both numerically and graphically in the computer lab, then back in the classroom a discussion of what we had seen and how this fit into the bigger picture we were developing.

Not everyone was happy with this format. Especially early in the first semester it went very slowly as students developed familiarity with the technology and learned a new way of approaching mathematics. A few students abandoned us at winter break. But by the second semester, the class was running smoothly and we were racing through the material. The high point came midway through the spring term when one of my students told me that she had just taken an engineering exam. She had forgotten the relevant formula for one of the questions. Then she thought about what she had learned in our class and figured out how to answer the question based on her knowledge of calculus. That level of understanding and ability to transfer is what I dream of for all my students. Ever since then, I have realized the importance of projects as a means of forcing students to explore and develop their own understanding of unfamiliar concepts. I have never again taught a course as project intensive as that year at Penn State, but I try to build at least three significant projects into every course I teach.

My last experience for this piece involved a second semester of real analysis at Macalester. In writing *A Radical Approach to Real* Analysis, I had read and come to appreciate Thomas

[2] If mistake it was. The argument can be made that Cauchy's understanding of convergence is what today we call uniform convergence. At the least, it demonstrates the difficulties inherent in ambiguous definition.

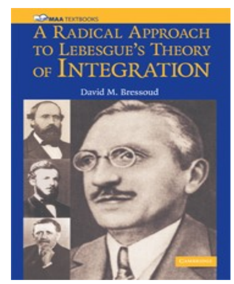
[3] I came to appreciate MathCAD, at least as it was configured in 1993, for the simplicity of the tools it made available. I have since taught calculus labs using *Mathematica* and *R*, the latter with purpose-built tools. Both suffered from what students perceived to be black boxes, removing an understanding of what the software was doing. That has never been as effective.



Hawkins's *Lebesgue's Theory of Integration: Its Origins and Development*. This is based on his doctoral thesis in the history of mathematics and presumes a reader who is already familiar with the subject. But it is so rich in describing the obstacles and difficulties that mathematicians encountered in the lead up to Lebesgue integration that I thought it would make for a fascinating text for this second semester of analysis.

There were eleven students enrolled in the course, and I ran it as a seminar. We started with a three-week crash course on the basics of measure theory using Bartle's *The Elements of Integration and Lebesgue Measure*. Then I turned the class over to the students, with two students responsible for presenting one or more sections of Hawkins at each meeting.

The course turned into a magnificent challenge for my students, so difficult that the entire class started meeting for an hour to an hour and a half *before* I met with them, so that they could try to come to an understanding of the material before they had to discuss it in seminar. At the end of the semester, one student described how much he had learned because this book constantly left him wrong-footed. Hawkins would describe a series of developments that in the next section would be revealed to be a blind alley, or at the least an incomplete understanding.



I later turned this experience into a textbook, *A Radical Approach to Lebesgue's Theory of Integration*. My goal, beyond providing an introduction to measure theory and Lebesgue integration, was to convey a sense of what it was like to explore and discover analysis in the latter 19^{th} century. I talk about the blind alleys and the incomplete understandings. Of all the books I have written, this is still my favorite.

When I began to teach, I thought that my role was to explain the mathematics as lucidly as I could, smoothing out the road for my students. What I have since learned is the importance of laying out challenges that stretch them, that force them into situations where they have to develop their ability to tackle unfamiliar material. The act of balancing challenges and supports requires fine-tuning with each new class, but I have learned that it is the essence of what it means to teach.

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