Connecting ordinary people with algebra

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I've been doing mathematics since around 1953 and have encountered with depressing frequency people who admitted, sometimes sheepishly but often with a certain pride, "math was my worst subject". When you ask questions, almost universally you find that their arithmetic knowledge is reasonably good (and, with some prompting, can usually remember how to add $\frac{1}{2}+\frac{1}{3}$) but they remember nothing about algebra. In my mathe-matical coffee table book *Indra's Pearls*, Dave Wright, Caroline Series and I defined our theoretical readers as those who "can handle high school algebra with confidence". Giving copies away to friends, I soon discovered these limitations made our audience quite small.

I began to try to understand this crazy situation almost 20 years ago. Why do I say crazy? Simply because a really huge amount of time and money is spent teaching a subject that most students promptly forget. I began to visit a lot of middle and high school classes, buy textbooks and to get to know some of the movers and shakers in the math ed biz. I think I now have a pretty good idea about what is going so wrong. The one liner is "the whole subject is taught with no apparent relevance to the students' lives". The subject is taught as pure math, not applied math. The applied parts, consisting of what are known as "word problems", are today the most hated part because these problems can't be solved by memorizing rules. But, more importantly, the word problems are not genuine interesting questions about the world but are always silly cooked up challenges whose answer is of no interest. Finally, almost every formula taught in algebra is expressed with the mysterious x and y, while in all real world formulas, *abbreviations* are used instead. Scientists, economists, statisticians, computer scientists etc. generally use x and y only for coordinates in geometry. It is true that pure mathematicians use them a lot, proving my point that school algebra has been highjacked by pure mathematicians.

We really need to stand back and ask: why is algebra actually useful and how might students be motivated to learn it? The answer is that in real life, one frequently encounters sets of numbers that, while the numbers vary from one instance to another, they *are always related by the same formula*. The best way to make this point is to discuss at length an example. I want to describe the simplest example, the example that ought to be every student's introduction to algebra, an example that shows that formulas are not scary, bizarre, esoteric objects, but friendly useful ways of understanding numbers you encounter regularly. Here it is:

$$\text{distance} = \text{speed} \times \text{time}$$

This holds for every trip, every motion, and every occurrence that takes place in space-time. This formula is (i) intuitive, (ii) powerful because if any two of the terms, distance, speed and time, are known, the third can be computed from the

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formula. Since the student sees easily how this is done, it becomes clear that the formula can also be written:

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{or} \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$

This rearrangement of the formula then seems intuitive rather being an ad hoc rule the teacher told the student to memorize. Next, like Einstein's famous formula $E = m.c^2$, where E stands for energy, m for mass and c, well, just a convention for the speed of light, any formula can be abbreviated using the first letters of the English words. In our example, this gives a concise version of the formula:

$$\mathrm{d} = \mathrm{s} \times \mathrm{t}$$

One can elaborate the formula too: suppose the trip is along a highway with mile markers. These markers create a coordinate for the one-dimensional road. Then, for a trip from B to A, one has:

$$\begin{aligned}&(\text{mile at point A}) - (\text{mile at point B})\\ &= \text{speed} \times ((\text{time passing A}) - (\text{time passing B}))\end{aligned}$$

The fact that formulas describe a numerical relationship that holds in as many cases as you want can be made even more clearly by the use of spreadsheets. In a spreadsheet, start by typing in the distances and speeds for many trips in 2 columns. Then enter only in the top cell of a third column not the time of this trip *but the formula for this time*, by referring the cells where its distance and speed are entered. (In the case of the top entries in the first two columns, this would be "= A1/B1".) Then drag the cell down and the same formula will be used on the whole column and you have a third column with the times for all the trips. This magical 21st century technique will surely wake up all the students!

Thus a formula is a way of encapsulating many specific instances in one handy universal form. Once understood, it is no more frightening than a sentence in English, a statement not of a relationship between a subject, object and verb but between a set of numbers. The formula is at your disposal all the time. For instance, how far away was that lightning strike? Multiply (1) the number of seconds between seeing the lightning (seen pretty much instantly at the time of the bolt) and hearing the thunder by (2) the speed of sound in feet per second. Another big lesson from this example: in the real world, formulas deal with numbers *expressed in units, seconds and feet in this example* and these units had better be used consistently. Traveling in Europe, by far the most important formula is:

$$\begin{aligned}&(\text{price in dollars})\\ &= (\text{exchange rate dollars per euro}) \times (\text{price in euros})\end{aligned}$$

where the units are euros and dollars. I think it fair to say that pure mathe-maticians hate units and take pains to avoid them; applied mathematicians know that they are indispensable.

This certainly is a good beginning but how should the rest of algebra be taught? The situation where using algebra clearly connects to middle and high school students is when money is involved. Wages, savings, prices are on everyone's mind. Again, spreadsheets are a major tool for examining budgets and seeing how changing one number affects the rest. A major topic for high school math should be compound interest. The way this grows often surprises people and many college graduates rue the day they took out that student loan without calculating carefully what salary they would need to pay it off. Struggling to find some place where polynomials were actually relevant, I wrote a blog post about compound interest, "How to get high school students to love a formula" [2] But, by and large, polynomials, factorization and solving quadratic equations are topics that can and should be dropped from the standard curriculum. They are about as useful to most students as Latin was a hundred years ago. Computer coding, business math with spreadsheets are topics to be added because they are very useful.

The same principle applies to geometry. Real problems, like finding the height of a tree or building from observations on the ground and using similar triangles, engage students. I taught a course at Brown for non-math majors and the second week I gave them the following: a photo of the Newport bridge taken from 18 miles up the Narragansett bay on a rise some 20 feet above water level, a photo of the bridge taken from a mile or so away and the height of the towers from which the bridge was suspended. I asked them: find the radius of the earth. I also gave them the diagram below showing how the curvature of the earth's surface means that part of the tower is below the horizon in the photo. The estimate came out fairly accurately.

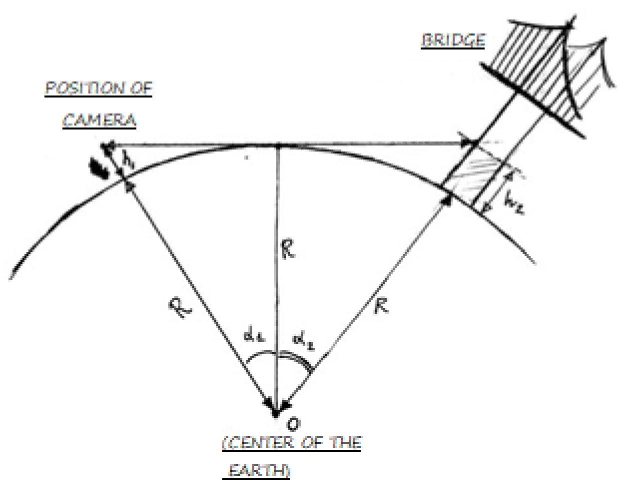

Figure 1: Given h_1 and h_2 (students found h_2 from the photos and the known height of tower) and the distance of the tower from the camera, find R. Caution: this problem leads to some nasty algebra which is easily avoided by dropping some very small terms. This is another big lesson in applying math to the real world: real data is not exact and the math dealing with it doesn't need to be exact either.

[2] http://www.dam.brown.edu/people/mumford/blog/2014/MiddleSchoolAlgebra.html.

I want to finish with some recommendations. I know school math has many many stakeholders and changing anything must be done incrementally. My belief is that *the applied math community needs to get more involved*. Their major professional organization is SIAM (Society for Industrial and Applied Mathematics) and it has an "activity group" on Math Education. But it does not seem to have national visibility. Statisticians and Computer Scientists have been signi cantly more involved recently in proposing new curricula that bring students into 21st century topics. They have promoted the teaching of basic probability/statistics and of computer coding – essential skills that will serve all students well.

The Common Core Standards are, without a doubt, the most influential recent proposal for revising K-12 curricula. The Math Common Core does have an emphasis on "modelling" that brings in applications but there's an issue here of the cart and the horse. What I have said is that one must begin with relevant applied topics and use them to lead the students to abstract ideas; the Math Core, however, starts with abstractions and allows teachers afterwards to embellish them with a model or two. The Math Core was apparently written primarily by three people (Daro, McCallum and Zimba) none of whom had any applied math background or K-12 classroom experience[3].These standards are being promulgated with a bunch of high minded laudable objectives. But how can you take these seriously when a large majority of people today refuse to read anything with a formula in it? For example, no general circulation magazine would dare to publish an article with a single formula in it. Unless one starts making the very first introduction to algebra more accessible and more intelligible and one follows through with continual attention to the relevance of each topic to the student's life and future career, their lofty goals will never be achieved.

My recommendation is simple: pay more attention to what is important, exciting and useful to the students themselves. Many people object that this results in "dumbing down" math and will not teach a certain unmeasurable but deep ability to think logically and abstractly. This feels to me like a specious justification, by those who mastered the rules of algebra, for the time and energy this required. Thinking deeply about any scholarly topic will teach a person to think logically. You don't need to torture students to master the art of factoring polynomials. But if we can teach more people to look at a formula as a kind of sentence that helpfully describes real and significant numerical relationships, I think the battle will have been won.

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[3] See https://seattleducation.com/common-core-standards/who-wrote-the-common-core-standards-the-common-core-24/ for a critical view.