Constrained by Unconscious Taboos:
The Case of Cross-Domain Problems

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Breaking rules; crossing boundaries

- Beginning geometry students sometimes get stuck because they unconsciously *assume* (without cause) that they are not allowed to modify a presented geometric figure by drawing auxiliary lines.
- A match-stick puzzle asks one to make, with six match sticks, four, non-overlapping, equilateral triangles. Many people get stuck on this problem since the solution requires three dimensions, to form a regular tetrahedron. They unconsciously *assume* (without cause), that the solution must remain on the surface on which the match-sticks are deployed.

Of course, related problem-solving activities can be designed to help students override these unconscious blocks. I want to discuss a kind of psychological block that I discovered among my (undergraduate) students that appears to stem from the conceptual organization of the curriculum. Specifically, the curriculum is organized into distinct topic domains, for example, number theory, algebra, geometry, trigonometry, calculus, etc. This makes sense, and has the advantage of developing a coherent and in-depth body of concepts, results, techniques, and perspectives in each domain. But a potential cost of this curricular geography is a loss of attention to synthesizing connections that link the various domains, that cross domain boundaries. I describe below one manifestation of this phenomenon.

A problem story

In one of my undergraduate courses I had introduced some basic combinatorics, and I was well into discussing some elementary number theory. I considered giving my students the following problem:

> Let N be an integer > 1. Call a divisor d > 0 of N a *separating divisor* if $gcd(d, N/d) = 1$. Find a formula for the number of separating divisors of N.

In fact, I decided to simplify the problem by giving them the answer, and just asking them to prove it.

> Let N be an integer > 1. Call a divisor d > 0 of N a *separating divisor* if $gcd(d, N/d) = 1$. Show that the number of separating divisors of N is 2^r, where r is the number of distinct prime divisors of N.

To my surprise, none of my students fully solved this problem, even though they had already demonstrated competence in all of the relevant mathematics. This led me to examine their work to seek some explanation.

Here is what I found. They had (correctly) 'type-cast' this problem as a number theory problem, and they had effectively mobilized number theory resources to analyze the situation. But then they could not finally see where the 2^r came from. To understand this, let's look at the situation in more detail.

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Bass, H. Constrained by Unconscious Taboos: The Case of Cross-Domain Problems. *WikiLetter 11, 31st July 2019.*

There are two versions of the Fundamental Theorem of Arithmetic. One, the *prime factorization*, says that N is a product of primes, uniquely up to the order of the factors. The other, the *prime power factorization*, says that $N = q_1 q_2 \bullet\bullet\bullet\ q_r$, where, for $j = 1, 2, \ldots, r$, $q_j > 1$ is a power of a prime p_j, and the p_j are distinct. Now let d be a separating divisor of N. If some p_j divides d, then all of q_j must divide d, since otherwise p_j is a common divisor of d and N/d, contrary to assumption. Thus d must be the product of a subset of the prime power factors $\{q_1, q_2, \bullet\bullet\bullet, q_r\}$. Finally, the number of subsets of this r-element set is 2^r, a fact already known to these students.

Notice that, in this final step, we have crossed a boundary from number theory into combinatorics, and this appears to have been the mind-set leap my students were unconsciously blocked from taking. I will refer to this kind of block as a *domain-separating wall*.

Cross-domain problems

The problem discussed above is an example of what I call a *cross-domain problem*, by which I mean a mathematics problem whose solution draws essentially on resources from two or more topic domains. Such problems can be mathematically rich, and, by their nature, they can evoke significant cross-domain connections. The lesson I drew from the episode described above is that: (1) Cross-domain problems have the potential to confront students with blocks arising from domain-separating walls; and (2) For this very reason, a natural way to intervene on this obstacle is to give students ample, scaffolded, opportunities to engage with cross-domain problems.

To implement this idea requires the construction of a rich supply of (non-superficial) cross-domain problems. Text books do not ease this task since, as a result, in part, of their organization aligned with the domain geography of the curriculum, they tend to contain few substantial cross-domain problems. This has led me to the belief that it would be a worthwhile effort for the mathematics education community to construct, and implement, a rich supply of grade-level appropriate cross-domain problems. I have found that this task can be both interesting and challenging. I will close now with some illustrative examples.

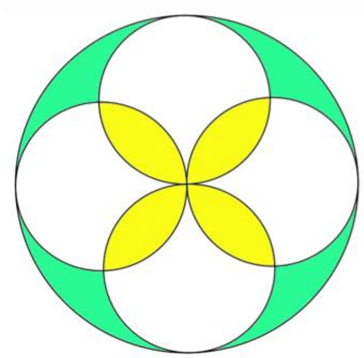
Some examples of cross-domain problems

1. **Rational approximation**. Suppose that a sequence of (rational) fractions approaches $\sqrt{2}$. Show that their denominators are unbounded.
2. **Divisibility**. Show that any product of d consecutive integers is divisible by $d!$.
3. **Place value**. (a) Exhibit a base $-$ 1,000 representation of:

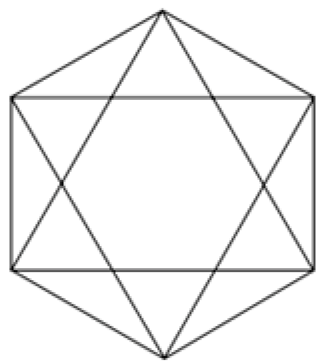
   N = 48,574,623,791,105

   (b) Recite the number N, and relate this to your answer.

4. **Geometric measure**. Which area is more, the green or the yellow?



5. **Geometric measure**. What fraction of the area of the big (regular) hexagon is the area of the interior hexagon?



6. **Higher derivatives of a product**. For a differentiable function, write Df for its derivative. The product rule says that the derivative of a product fg is given by $D(fg) = (Df)g + f(Dg)$. Develop a formula for the n^{th} derivative of fg: $D^n(fg)$.

Discussions of the problems

1. **Rational approximation**. If their denominators were bounded, say by d, then they could all be put over the common denominator $D = d!$. Hence the sequence consists of integer multiples of $1/D$, so no two of them can get closer than $1/D$. Hence a convergent sequence of such fractions must be eventually constant, and so have a rational limit. But $\sqrt{2}$ is irrational.

2. **Divisibility**. For whole numbers d and n, the binomial coefficient, $\binom{n}{d} = \text{“n} - \text{choose} - \text{d,”}$ is the number of d-element subsets of an n-element set; in particular, it is an integer. From combinatorics, it is given by the formula $\binom{n}{d} = \frac{n(n-1)(n-2)\bullet\bullet\bullet(n-d+1)}{d!}$. The numerator in this expression is a product of d consecutive integers, descending from n.This proves the claim for $n \geq 0$. For $n = -m < 0$, we have $\frac{n(n-1)(n-2)\bullet\bullet\bullet(n-d+1)}{d!} = (-1)^d \frac{m(m+1)(m+2)\bullet\bullet\bullet(m+d-1)}{d!} = (-1)^d \binom{m+d-1}{d}$, which is again an integer.

3. **Place value**. (a): The point is to recognize that, *one is staring at an answer*. The commas separate at-most-three-base-10-digit-numbers (0 through 999), and these are exactly the base-1,000 “digits.”

We can write, $N = 48 \bullet 1{,}000^4 + 574 \bullet 1{,}000^3 + 623 \bullet 1{,}000^2 + 791 \bullet 1{,}000 + 105.$
(b): Reciting N, we say: "48 **trillion,** 574 **billion**, 623 **million**, 791 **thousand**, 105."
Note that: million = $1{,}000^2$; billion = $1{,}000^3$; and trillion = $1{,}000^4$. And so, *we are actually speaking in* base $- 1{,}000$!

4. **Geometric measure**. Let G be the green area and Y the yellow area. The white circles have half the diameter, hence ¼ the area of the big circle. Let W be the area of the union of the four white circles, and C the area of the big circle. Then $W = 4(C/4) - Y = C - Y$, while $G = C - W = C - (C - Y) = Y$.
5. **Geometric measure**. Here is one approach. Using rotational symmetry, we see that the big hexagonal area, H, is partitioned into: E = a set of six congruent isosceles triangles, each with an edge on the boundary; V = a set of six congruent equilateral triangles, each with a vertex on the boundary; and h = the interior hexagon. The triangles in E and in V have equal areas (equal base and equal height). We can partition h into W = a set of six congruent equilateral triangles by connecting its center to each of its vertices. It can be seen that the triangles in W are reflections of, and hence congruent to, those in V. Thus, H is partitioned into a set $(E \cup V \cup W)$ of 18 triangles, all of equal area, and 6 of which (W) partition h. And so, the area of h is $6/18 = 1/3$ of the area of H.
6. **Higher derivatives of a product**. $D^0(fg) = fg$

$$D^1(fg) = (Df)g + f(Dg)$$

$$D^2(fg) = [(D^2f)g + (Df)(Dg)] + [(Df)(Dg) + f(D^2g)] = (D^2f)g + 2(Df)(Dg)] + f(D^2g)$$

$$D^3(fg) = [(D^3f)g + (D^2f)(Dg)] + 2[(D^2f)(Dg) + (Df)(D^2g)] + [(Df)(D^2g)] + f(D^3g)]$$

$$= (D^3f)g + 3(D^2f)(Dg) + 3(Df)(D^2g)] + f(D^3g)$$

In general, we can see that $D^n(fg)$ has the form.

(*) $D^n(fg) = \sum_{0 \le d \le n} B(n,d)(D^d f)(D^{n-d}g)$, for some coefficients $B(n,d)$ $(0 \le d \le n)$, with $B(0,0) = 1$; $B(1,0) = 1 = B(1,1)$; $B(2,0) = 1 = B(2,2), B(2,1) = 2$; and $B(3,0) = 1 = B(3,3)$, and $B(3,1) = 3 = B(3,2)$. These numbers strongly suggest the Binomial Coefficients. Differentiating (*) we obtain, $D^{n+1}(fg) = \sum_{0 \le d \le n+1} B(n+1,d)(D^d f)(D^{n+1-d}g) = \sum_{0 \le d \le n} B(n,d)[(D^{d+1} f)(D^{n-d}g) + (D^d f)(D^{n+1-d}g)] = \sum_{0 \le d \le n+1}[B(n,d) + B(n,d-1)](D^d f)(D^{n+1-d}g)$,

where, for $d = 0$, we agree that $B(n,-1) = 0$. It follows that,

$$B(n+1,d) = B(n,d) + B(n,d-1)$$

This is the analogue of the Pascal Relation for Binomial Coefficients, and so, it follows by induction that, in fact, $B(n,d) = \binom{n}{d}$.

It is natural to ask whether this result could be directly derived from the binomial Theorem. The only way I have found to do this uses functions of two variables, and partial derivatives.