From Arithmetic to Algebra, Part 1: Algebra as Generalized Arithmetic*

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The school mathematics curriculum is an organic entity: it grows from the simple to the complex, from the concrete to the sophisticated and abstract. It cannot be otherwise because its main function is to provide guidance to students' first tentative steps in the learning of mathematics and, lest we forget, the logical structure of mathematics itself also grows from the simple to the complex.

Unfortunately, there are unnecessary discontinuities in the development of the school mathematics curriculum[2] that disrupt student learning, and one result of this disruption has led to fear and apprehension in the learning of algebra. In this WikiLetter, we will focus on the discontinuity from arithmetic to algebra. Here, "arithmetic" refers to whole numbers, finite decimals, and fractions, and "algebra" refers to *introductory* school algebra, i.e., rational numbers, the extensive use of symbolic notation, linear equations of one or two variables, and quadratic equations. We will omit any reference to the more advanced topics of school algebra such as exponential functions, logarithms, and the formal algebra of *polynomial forms* (see Schmid-Wu, 2008, Wu, 2016b, and Wu, *to appear*), and will, instead, focus on the two major characteristics of school algebra: generality and abstraction. Rather than indulging in the rather futile and time-consuming exercise of explaining the meaning of the latter concepts, we trust that their meaning will emerge with clarity in the course of the ensuing discussion.

The main message of this article is that school algebra is just *generalized arithmetic*. Now, this is a slogan that probably means different things to different people. It therefore behooves us to explain—as precisely as possible—what *generalized arithmetic* means in the present context. Arithmetic is concerned with accurate computations with specific numbers. In the way school mathematics is usually taught, children's main concern in arithmetic is with the here and now, in the sense that students are content with the correctness of getting the answers to, e.g., $151 - 67$ or $\frac{5}{6} + \frac{3}{8}$ as 84 and $\frac{58}{48}$, respectively, but not much beyond that. Algebra is, however, students' first contact with mathematics proper and it brings to students an awareness of "the big picture". It asks students to begin thinking about whether a given computation is part of a general phenomenon, or whether they can put a given computation in a broader context to understand it better. But for arithmetic students who have been happily computing away, they have to wonder *why* they should switch gears to learn this *generalized arithmetic*. Why bother, indeed?

Any mathematics curriculum that is serious about smoothing students' passage from arithmetic to algebra will have to answer the last question. What might the answer be?

We will discuss two topics in algebra, summing a finite geometric series and solving a linear equation, by way of answering this question. In the process, we also hope to clarify the nature of generalized arithmetic. We forewarn the reader that this discussion will involve some precise mathematics, something not commonly found in writings of this kind. We do so because it is necessary, and also because the heart of any discussion in mathematics education usually lies in the mathematics itself.

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[2] We want to be explicit that we have in mind mainly the discontinuities in the American *school curriculum* rather than any inherent discontinuities in school mathematics itself.

Suppose we ask 8th-graders to compute (calculators allowed): $1 + 3 + 3^2 + 3^3 + \cdots + 3^9 + 3^{10}$
As arithmetic, the answer is 88,573, and that is all there is to it. But does this computation stand alone, or is this part of a general phenomenon? The eighth-graders should first take note of the unusual *pattern*, or *structure*, in this sum: the increasing powers of 3 (assuming they know $1 = 3^0$ and $3 = 3^1$). The first thought that should cross their minds is that they should be able to get the sum of increasing powers of 3 up to, not just 10, but any integer. In fact, why 3? Why not the sum of increasing powers of an arbitrary number up to any integer? Right away, this very question has left arithmetic behind because, trying to add the increasing powers of an *unknown number* up to some *unknown integer* is to give up any hope of computing with explicit numbers. In fact, we have to confront a more fundamental problem even before we get started: how to express this kind of *generality* (sum of increasing powers of an *unknown number* up to an *unknown integer*) in a concise way. At this point, we have to tell eighth-graders that, for more than a thousand years, humans struggled mightily with the issue of how to handle such *generality* until around 1600, when they finally found an efficient way to use symbols for this purpose. So carrying on this mathematical legacy, we let r be an arbitrary number and let n be an arbitrary positive integer and then ask for the sum of $1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n$. Of course, by giving up on getting explicit numbers at each step, the only way to do any computation is to rely on the associative and commutative laws of + and ×, and the distributive law which are applicable to *all* numbers, known or unknown. Such a computation will finally allow eighth-graders get to see the importance of these laws of operations, perhaps for the first time.

Instead of plunging headlong into the new world of generality, it may be prudent to first go only partway and try to get the sum of $1+3+3^2+3^3+\cdots+3^{23}+3^{24}$. This is a sum of 25 explicit numbers, so it looks like an ordinary arithmetic problem. But because getting this sum by brute force computation will require more patience than most eighth-graders can muster, it is clear that something more than ordinary arithmetic skills will be needed to get an answer. One way to proceed is the following. We look at this sum as one number regardless of the fact that we do not know what it is explicitly and denote it by S. Right away we are capitalizing on the advantage of using symbols. This is a key step, because, at this point, we are no longer doing the arithmetic of old by insisting on an explicit answer for every computation at every step. Rather, we will compute with S simply *as a number*. That said, since S is the sum of increasing powers of 3, the idea of trying to multiply S by 3 almost suggests itself because, by applying the distributive law, we immediately get a sum that is "almost the same" as S itself. Precisely, we get: $3S = 3 \cdot 1 + 3 \cdot 3 + 3 \cdot 3^2 + \cdots + 3 \cdot 3^{23} + 3 \cdot 3^{24} = 3 + 3^2 + 3^3 + \cdots + 3^{24} + 3^{25}$
$= (1 + 3 + 3^2 + 3^3 + \cdots + 3^{24}) - 1 + 3^{25} = (S - 1) + 3^{25}$. Therefore, **(1) $3S = S + 3^{25} - 1$.**
Now, still without knowing the precise value of S, we add $-S$ to both sides to get: $(-S) + 3S = (-S) + S + 3^{25} - 1$. After one more application of the distributive law on the left side, we get $(-S) + 3S = (-1 + 3)\,S = 2S$, so that **(2) $S = \frac{1}{2}\left(3^{25} - 1\right)$.** With the help of a calculator, we get S = 423, 644, 304, 721, and we should not fail to point out to eighth-graders that the answer has been obtained without any sweat.

This is the first fruit of *abstract thinking*: an answer obtained not by brute force computations with explicit numbers but by injecting reasoning and abstract pattern-recognition into computations with numbers in general. For example, the decision to multiply S by 3 to get to equation (1)—which is as good as the final answer (2) itself—was prompted by the reasoning that the abstract pattern of a sum of powers of 3 will essentially repeat itself after this multiplication. In one sense, what we have done is just arithmetic because it is nothing more than the application of arithmetic operations on numbers, *and numbers only*. At the same time, it also goes beyond arithmetic because this is a computation with unknown numbers using only the laws of operations. We are doing *generalized arithmetic*. What we have just learned is that, by stepping away from computations with explicit numbers and by trying to discern patterns on a larger scale, we have much to gain even in ordinary computations. We can now return to our original problem of getting the sum of $\boldsymbol{1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n}$ **for any $\boldsymbol{r}$ and any positive integer $\boldsymbol{n}$.** Observe that if $r = 1$, this sum is equal to $n + 1$ and is of no interest. So we assume $r \neq 1$ from now on. If we retrace the preceding reasoning with care, we can see without difficulty that it remains valid almost *verbatim* if we replace 3 by r and the exponent 24 by n. Thus, if $r \neq 1$, and if n is a positive integer, then

(3) $\boldsymbol{1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n = \dfrac{r^{n+1} - 1}{r - 1}}$**.** This is the so-called *summation formula for a finite geometric series*. Equation (2) is the special case of equation (3) when $r = 3$ and $n = 24$. To fully savor this kind of generality, we notice that identity (3) is also fit for getting the sum: $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{27}} + \frac{1}{2^{28}}$.

Now we can ask the same questions again about identity (3): is this part of a general phenomenon or is it an isolated result? Can we put it in a broader context to understand it better? Indeed, we can understand it better, but to get the "broader context" we will have to cast a wider net. All eighth-graders know the well-known identity $(x - y)(x + y) = x^2 - y^2$ for all numbers x and y. It is a simple exercise in the use of the distributive law to prove the following more general identity: for any positive integer n and for all numbers x and y,

(4) $\boldsymbol{(x - y)(x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} + y^n) = x^{n+1} - y^{n+1}}$**.** Thus $(x-y)(x+y) = x^2-y^2$ is a special case of identity (3) when $n = 1$. Moreover, identity (3) is also the special case of identity (4) when $x = r$, $y = 1$, and $r \neq 1$. In particular, identity (3) on summing a finite geometric series is now seen to be related to the mundane identity $(x-y)(x+y) = x^2-y^2$. More is true, however. Consider 64,339,280,491, which is a big number. It is not easy to tell whether 64,339,280,491 is a *prime*, i.e., a number divisible only by 1 and itself. But it happens to be equal to $31^7 - 4^7$, so we know from identity (4)—with $x = 35$, $y = 4$, and $n = 6$—that 31 divides it because, 64, 339, 280, 491 = $31^7 - 4^7 = 31 \times (35^6 + 35^5 \cdot 4 + \cdots + 35 \cdot 4^5 + 4^6)$. It is not a prime! Similar considerations show that if x and y are positive integers and $x - y > 1$, then for any $n \geq 1$, $x^n - y^n$ is never a prime. Identity (4) now opens up a completely new area for discussion about prime numbers (see Section 1.3 of Wu, 2016b for more details). What is noteworthy is that while we are still doing arithmetic, we discover that, by introducing symbols to represent numbers and by welcoming the concept of generality into our computations, the consideration of the summation of a finite geometric series is seen, via identity (4), to be related to the question of whether some whole numbers are primes. This is an example of the power of generality that eighth-graders can appreciate.

Next, consider **the solutions of equations**, a topic central to any school algebra curriculum and—in fact—to the historical development of algebra itself. We can make our point with a simple linear equation, $4x+1 = 2x-3$. In the usual presentation of school algebra, x is a *variable* and we solve the equation by the following symbolic manipulations: **Step A:** $(-2x) + 4x + 1 = (-2x) + 2x - 3$; **Step B:** $2x + 1 = -3$;
Step C: $2x + 1 + (-1) = -3 + (-1)$; **Step D:** $2x = -4$; **Step E:** $x = -2$.

The answer of -2 is correct, but a little reflection would reveal that *these five steps make no sense whatsoever*. Consider Step A, for example. Since x is a quantity that *varies*, what does it mean to say that the two quantities that vary, $4x+1$ and $2x-3$, are somehow "equal"?[3] And how do $4x+1$ and $2x-3$ stay being "equal" after the varying quantity $-2x$ has been added to both? Moreover, the passage from Step A to Step B requires that we apply the distributive law and the associative law of addition to both $(-2x) + 4x + 1$ and $(-2x) + 2x - 3$. Now

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[3] For example, since x can vary, x may be equal to 0. In that case, $4x + 1 = 1$ and $2x - 3 = 3$. How can $1 = -3$?

students learned in arithmetic that these laws of operation are applicable to numbers, but where is it *explained* that these laws are equally applicable to *quantities that vary*? And so on. Such a drastic turn of events, from explicit computations to unknowable computations, can be disorienting to students. Is mathematics supposed to teach students how to reason, or is it supposed to encourage rote-learning?

This is but one of many blatant examples of how the usual school curriculum disrupts students' transition from arithmetic to algebra. The main culprit is the cult of "variables" and its inevitable subsequent abuses. We can rectify this unwarranted disruption by restoring the idea of algebra as *generalized arithmetic*, as follows.

First, we have to come to terms with what an "equation" is. The meaning of the equation $4x + 1 = 2x - 3$ is that it is a *question*: is there a number x so that $4x + 1 = 2x - 3$? Such a number x is then called a **solution** of $4x + 1 = 2x - 3$. Note that no "variable" is involved. Just numbers. The way to *solve* this equation is to assume that there is a solution x. Remember: this x is *now* a single number. So we have the equality of two numbers, $4x + 1 = 2x - 3$, and Steps A to Step E now become five successive statements about numbers, each step being a consequence of the preceding one. At the end, what we get is this: *If* there is a solution x of $4x + 1 = 2x - 3$, this solution has to be -2. This does *not* say that -2 is a solution of $4x+1 = 2x-3$. To prove that, we must substitute $x = -2$ into $4x + 1 = 2x - 3$ to check that the two sides are indeed equal, which is easily accomplished. In this light, once Steps A to E are properly interpreted as computations with numbers, known and unknown, they are actually correct! Thus, solving equation (any equation, as it turns out) is part of *generalized arithmetic*.

We emphasize once again that no "variable"[4] is involved in the solution. (See Section 3.1 of Wu, 2016b for a more comprehensive discussion of these issues.)

We hope the preceding discussion has given some indication of why we say that school algebra is *generalized arithmetic*: it is arithmetic on a more abstract level. It is incumbent on the school curriculum to help students bridge the gap between arithmetic and algebra by introducing abstraction into the arithmetic curriculum when feasible, by making use of symbols when the occasion calls for it, and, above all, by making reasoning a daily routine in arithmetic because all of mathematics—at *any* level—is built on reasoning. The failure of the usual school curriculum in the U.S.[5] to meet these basic requirements accounts for the discontinuity in the school curriculum from arithmetic to algebra that is so detrimental to the learning of algebra. In Part 2, WikiLetter 10, we address how to teach arithmetic better.

References

[1] Common Core (2010). *Common Core State Standards for Mathematics*. Retrieved from: http://www.corestandards.org/Math/

[2] Schmid, W. and Wu, H. (2008) The major topics of school algebra (March 31, 2008). https://math.berkeley.edu/~wu/NMPalgebra7.pdf

[3] Wu, H. (2016b). *Teaching School Mathematics: Algebra*. Providence, RI: American Mathematical Society. Its ***Index*** is available at: http://tinyurl.com/haho2v6

[4] Wu, H. (to appear). *Rational Numbers to Linear Equations*, *Algebra and Geometry*, and *Pre-Calculus, Calculus, and Beyond*.

[4] Let it be noted that *variable* is not a mathematical concept, period. If students are taught to use symbols correctly, they will have no need to worry about what a "variable" means. See Section 1.1 of Wu, 2016b.

[5] As of 2018, most schools in the U.S. are trying to adjust to a new curriculum mandated by the Common Core Standards (see Common Core 2010). The outcome is uncertain.