# From Arithmetic to Algebra, Part 2:
# How to Teach Arithmetic Better*

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In Part 1, WikiLetter 9, we pointed out that the main focus of arithmetic in the school curriculum has been on accurate computations with specific numbers. While introductory school algebra is also concerned with computations with numbers, it computes with known and unknown numbers alike—relying only on the laws of operations (associative and commutative laws of + and × and the distributive law)—and begins to look for abstract patterns in numbers that are true for numbers in general. Now in Part 2, WikiLetter 10, we will consider students' difficulty in making the transition from arithmetic to algebra, and more importantly, how to deal with the difficulty.

The *usual school curriculum* fails to address this difficulty. Some educators became aware of the difficulty of this transition and have come to advocate the introduction of "algebraic thinking" in the elementary grades, e.g., Blanton, 2018, Kaput, 2008, and Kieran, 2004. Their intentions cannot be faulted, but as in all things in mathematics education, good intentions are not enough because the devil lurks in the details. Given the fact that the present school algebra curriculum in the U.S. is very seriously flawed (cf. Wu, 2016b, especially Sections 1.1, 2.1, 4.3, 5.1, 7.2, 8.4, and 10.4),[2] one has to first find out what this "algebraic thinking" is all about in the context of such a defective curriculum because, "What we think algebra is has a huge bearing on how we approach it" (Kaput, 2008, p. 8). In addition, the general recommendations for achieving this goal usually involve pedagogical embellishments and the introduction of new elements in arithmetic instruction (such as thought-provoking problems) while seemingly leaving unperturbed the existing defective arithmetic curriculum.

Our belief is that a simpler approach, one that sharply focusses on teaching *correct* arithmetic in elementary school would be more effective in bringing about improvement. One should not mistake this belief to mean that we are pursuing the teaching of correct arithmetic as an end in itself (although there is no denying that it is a laudable goal). The virtue of correct arithmetic is that it provides the right platform from which to launch algebra, as the following recommendations (1)-(7) will make this point abundantly clear. Correct arithmetic also happens to be *more learnable* than defective arithmetic, and it goes without saying that better-informed arithmetic students will be in a better position to learn algebra.

We will eschew generalities in the following discussion. In particular, we will not engage in making *broad* suggestions on how to reconceptualize parts of the arithmetic curriculum. We will, instead, get down to the fundamentals by suggesting specific changes in the *mathematics* taught in the arithmetic curriculum, and the specificity is made possible by the ability to reference the following six volumes[3] by chapter and verse: Wu, 2011, 2016a, 2016b, and *to appear*. (We may add that it is precisely with the goal of achieving a wholesale change in

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[2] The same can be said about the American school mathematics curriculum as a whole at the moment; cf. Section 2.3 and Appendix 2 of Wu, 2018. However, it would be naive to assume that U.S. is the only country that is struggling with school mathematics education.
[3] The early drafts (Wu, 2000, 2001, 2010a, and 2010b) are free and online.

the *mathematical content* of school mathematics that these six volumes have been written.[4]) That said, we will now indicate several key improvements[5] that we believe should be made in the teaching of arithmetic to smooth students' transition from arithmetic to algebra.

(1) *Mathematical* explanations (in addition to pictures and analogies) should be given for the standard algorithms in a grade-appropriate manner, including the emphasis on the importance of the associative and commutative laws of addition for the standard algorithms for addition and subtraction (e.g., as explained on page 68 of Wu, 2011 or pp. 46 and 49 of Wu, 2000), and the crucial role played by the distributive law in the multiplication algorithm and the long division algorithm (e.g., page 85 of Wu, 2011 or pp. 62, 66, and 82 of Wu, 2000). When students get to know the reasoning behind the algorithms, they can make better sense of the algorithms as well as these laws of operations (e.g., as explained on Chapter 2 of Wu, 2011 or Section 2 of Wu, 2000). At present, one reason these laws are not taken seriously by many teachers and students is that the existing curriculum does not put them to use in a mathematically substantive way to make any real impression on students. It therefore comes to pass that these laws are regarded as nothing more than things to memorize for acing standardized tests.

(2) Even in arithmetic, students can begin to learn about abstractions and structure. Indeed, the overriding theme of the four standard algorithms is that *a knowledge of single-digit computations empowers us to compute with any whole numbers, no matter how large* (see Chapter 3 of Wu, 2011, and it is of course repeated ad nauseam all through Chapters 4-7, *loc. cit*.; also see pp. 38-40 of Wu, 2000). If we remind elementary students of this fact, and do it often enough, then they would not only understand the reason these algorithms are worth learning, but also become more familiar with abstract thinking and less likely to be shocked in their confrontation with algebra. (In fact, if this overriding theme were forcefully brought out by teachers in their teaching, they might be more successful in persuading students to memorize the multiplication table.)

(3) In the same vein, the overriding importance of the theorem on equivalent fractions (Chapter 13 of Wu, 2011, or Section 1.3 of Wu, 2016a; see also Section 3 of Wu, 2001) should be impressed on students, including its direct impact on the comparison (ordering) of fractions (pp. 31-35 of Wu, 2016a or Section 5 of Wu, 2001), the addition and subtraction of fractions (Section 1.4 of Wu, 2016a or Section 6 of Wu, 2001), and the multiplication and division of fractions (Sections 1.5 and 1.6 of Wu, 2016a or pp. 33-37 and 72 of Wu, 2010a). Without this understanding, students do not see equivalent fractions as the abstract unifying theme that connects all the above diverse skills. Rather, they think of fractions as a fragmentary subject and come to believe that the only reason for having this theorem is for simplifying fractions.

(4) Finite decimals should be defined as a special class of fractions (the *decimal fractions*) and taught as such (see Section 12.3 of Wu, 2011 or pp. 20-22 of Wu, 2010a). This approach is both historically and pedagogically correct (*loc. cit.*), and it is only from this vantage point that the four arithmetic operations—especially multiplication—on finite decimals can be made transparent and transparency is clearly a prerequisite for learnability. (See Section 14.2, p. 256, p. 269, and Section 18.4 of Wu, 2011; see also pp. 48-49, 65-66, and

[4] Precisely, the goal is to eradicate what we call Textbook School Mathematics (TSM) from K-12 mathematics education altogether (see Section 2.3 and Appendix 2 of Wu, 2018 for an explanation of TSM), and the above-mentioned six volumes show how this could be done by giving a complete and coherent exposition of the mathematics of the K-12 curriculum that is grade-appropriate, and equally importantly, mathematically correct.

[5] Because this is a short paper, some of the important topics in school arithmetic fall outside of the paper, most notably, the notions of *constant speed* (Section 7.2 of Wu, 2016 b) and *slope* (Section 4.3 of Wu, 2016, b).

76-79 of Wu, 2010a). This is another opportunity for students to appreciate mathematical abstraction and structure when two kinds of seemingly different numbers are revealed to be basically one and the same.

(5) The parallel between the arithmetic operations on whole numbers and those on fractions should be stressed (pp. 173-174, 221, 262, and 284-286 in Wu, 2011, or pp. 46, 63, and 81-82 of Wu, 2001). The fact that this parallel enhances the learnability of fractions is too obvious for comment. Less obvious but no less important is the fact that, by emphasizing this parallel, we can reinforce students' appreciation for abstraction and structure: what they learn in whole numbers will help them learn fractions because these are similar topics. It used to be believed (*perhaps* less so in the last few years with the advent of the Common Core State Standards for Mathematics (see Common Core, 2010)) that "fractions are such different numbers from whole numbers", and this false belief has naturally hampered student learning in fractions.

(6) The teaching of fractions should respect—in a grade-appropriate manner—the abstraction and generality that are inherent in the subject. There is now something close to a consensus in the U.S. (see, e.g., Common Core, 2010) that, for example, giving a fraction a precise definition as a point on the number line (Jensen, 2003 and Wu, 2001) is more pedagogically effective than making believe that a fraction is simultaneously a piece of pizza, a division, and a ratio. If we were to make an effort to state the various basic facts about fractions concisely, then students would naturally (and gradually) learn about generality and become familiar with the use of symbols. There are ways to do this without sacrificing mathematical correctness, e.g., Jensen, 2003, or Part 2 of Wu, 2011.[6] For example, the theorem on equivalent fractions is the statement that for *any* fraction $\frac{m}{n}$ and for *every* nonzero whole number $c$,

$$\frac{m}{n} = \frac{cm}{cn}$$

Likewise, the cross-multiplication algorithm (one of the most important skills in fractions) is not "the butterfly" but the statement that, for *any* two fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, $\frac{m}{n} = \frac{k}{\ell}$ is equivalent to $m\ell = nk$ (in the process, of course students will also learn what the phrase "is equivalent to" means). The formula for the addition of two *arbitrary* fractions $\frac{m}{n}$ and $\frac{k}{\ell}$ is

$$\frac{m}{n} + \frac{k}{\ell} = \frac{m\ell + kn}{n\ell}$$

And so on. Such exposure to generality and the use of symbols smooths students' passage to algebra.

(7) Finally, *we should strive to provide reasoning for every claim in arithmetic*. The need for doing this can be easily understood when one realizes that, whereas computations with specific numbers in arithmetic can be accomplished (at least on a superficial level) by the memorization and execution of rote skills, it is much more difficult to learn algebra by rote memorization because—as we pointed out about generalized arithmetic—the key issues in algebra are about the truth of statements concerning numbers *in general* (most often for *all* numbers). Reasoning becomes the only vehicle for navigating the terrain of algebra and we are therefore obligated to acclimate students to algebra by exposing them to the use of reasoning from day one. For example, every statement in Wu, 2011, or Wu, 2016a and 2016b is supported by reasoning.

We have made an effort to explain in some detail the meaning of introductory algebra as *generalized arithmetic* and how we might improve the arithmetic curriculum to facilitate students' transition from arithmetic

[6] The treatment of fractions in Chapter 1 of Wu, 2016a, is slightly more sophisticated, but we have to mention it because it is one that happened to serve as a blueprint for the Common Core Standards.

to algebra. We do not pretend that achieving such improvement will be easy as it involves sustained professional development for teachers and the creation of reasonable textbooks for students. But we *must* try.

One thing is clear though. The strategy of banishing abstraction from arithmetic is designed to delude students into thinking that they can win battles and skirmishes in their march through computations without having to do any abstract thinking. Unfortunately, it is this strategy of avoidance that causes students to lose the war by the time they get to algebra, where they get shell-shocked when confronted with abstraction, generality, and the extensive use of symbols. If we do not improve the arithmetic curriculum, we also lose the same war.

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