The Negative View of Proof

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Proof is central to mathematics, and consequently to mathematics education. However, many researchers have noticed that students have great difficulties in understanding when an argument can justifiably be called a proof, and when it must remain merely an argument (e.g. Harel & Sowder, 1998). But students are not the only group who struggle with this problem: characterising when an argument is viewed as a proof, and identifying the processes used by mathematicians to reach this judgement, are difficult philosophical problems. In this article, firstly, I outline two existing approaches to addressing the problems. Then, I propose a new approach, which I refer to as the negative view.

A proof is a valid argument

An obvious starting point in the search for a characterisation of proof is to adopt a formalist position and assert that a proof is merely a valid mathematical argument with reference to some background theory (e.g. Mariotti, Bartolini Bussi, Boero, Ferri & Garuit, 1997). But if validity is to be a successful method of characterising proof, then it must be defined. The traditional formalist approach is to define a valid proof as a sequence of formulae, each of which is either an axiom or follows from earlier formulae by agreed rules of inference. This approach fails, however, when it is tested against mathematical practice: proofs in mathematics journals and textbooks are just not like this.

Azzouni's (2004) derivation-indicator view is an attempt to deal with this problem. He highlighted the distinction between 'derivations' (which do start from axioms and do spell out in detail every logical inference), and 'proofs' (arguments of the type found in mathematics journals and textbooks). Azzouni accepted that most proofs are not derivations, and so are not valid in some technical sense. Instead, he suggested that an argument can be called a proof if it successfully indicates the existence of an underlying valid derivation. And a derivation is valid if each and every step in its logical chain legitimately follows from a subset of the earlier steps together with the axioms (whether this is the case is, in principle, mechanically checkable).

Azzouni's suggestion does not explain how mathematicians do or could reach this judgement: how does one decide whether a given argument successfully indicates the existence of a derivation? Given the limited experience that mathematicians have with derivations, and their extreme length, it seems unlikely that any psychologically plausible account of proof could explicitly involve them (Pelc, 2008). The case is even weaker for the case of students, who encounter 'valid' proofs before they encounter the idea of derivations (if they encounter this idea at all). How could a student ever determine whether an argument indicates the existence of a derivation? One possible answer is to assert that neither mathematicians nor students explicitly evaluate whether a derivation is indicated by an argument, but use other methods that are functionally equivalent to doing so.

But this solution raises an obvious question: if some unspecified process (not explicitly related to derivations) can be used to determine whether a derivation is indicated, then (a) what is this process? And (b) wouldn't it be more parsimonious and psychologically plausible to characterise proof as being an argument which successfully passes the test of this unspecified process, not as an argument which indicates a derivation?

A proof is a convincing argument

The second broad class of approaches to the problem of characterising proof relies upon conviction. Put simply, under this view a proof is any argument that convinces an appropriate audience (e.g. Balacheff, 1987; Davis & Hersh, 1983; Hanna, 1991). The actual processes used by

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the audience to determine whether they are convinced may vary substantially (e.g. Harel & Sowder, 1998).

This characterisation moves the focus of the question away from the argument itself and towards the reader. If the reader is convinced, then the argument is a proof, if the reader is not, then it is not. But this feature, while appealing to some, is a weakness to others. Selden and Selden (2003), for example, pointed out that the validity of a proof appears to be independent of the reader: "Mathematicians say that an argument proves a theorem, not that it proves it for Smith and possibly not for Jones" (p. 11). Azzouni (2004) observed that mathematicians are "good at agreeing with one another on whether some proof convincingly establishes a theorem" (p. 84), and suggested that these high levels of agreement are problematic for the conviction account.

A second problem for this account is that mathematicians are convinced by arguments that they would not wish to characterise as proofs. For example, empirical investigations of the Riemann Hypothesis have presented an extremely convincing case for its correctness (du Sautoy, 2004), and heuristic arguments about the falsity of the Continuum Hypothesis are similarly persuasive (to some at least, Freiling, 1986).

It is possible to supplement the conviction criterion with additional criteria. Harel and Sowder's (1998) 'proof schemes' theory took this approach: while they placed conviction at the heart of proof, they also emphasised that students must be encouraged to use only the proof scheme "shared" by modern mathematicians, which they interpreted to be valid deductive arguments. Thus this approach avoids the requirement to explain why convincing non-deductive arguments are not proofs, but at the cost of requiring a solution to the question addressed to the validity approach earlier: what determines whether a deductive argument is perceived as being valid or invalid?

A negative view

The two approaches I have outlined above share a common feature: they provide positive characterisations of what a proof is or should be (it should be valid, it should be convincing etc). But the American essayist Neil Postman (1992) pointed out that sometimes difficult concepts are more productively defined negatively. I briefly discuss two examples here: health and justice.

The medical profession is concerned with health, but what does this mean? When is a person healthy? Providing a positive characterisation of health is extremely difficult, and sensible doctors do not attempt to do so. Instead, they evade the question by asserting that a patient is healthy if she is not ill. Illness, in contrast to health, is straightforward to define positively: a person is suffering from illness X if they have symptoms Y that are caused by Z. What consequences does this negative approach have for health judgements? I will outline several.

When a person has a serious condition, there will be few disagreements between expert doctors over their health (no one displaying the symptoms of chronic high blood pressure will be characterised as healthy by their doctor). In contrast, when a person has a less serious condition (perhaps they have a bad sore throat), if forced to categorise them as "healthy" or "ill", some doctors will choose one category, some the other. In other words, when the person is very ill the illness/health judgement is objective (in the sense that it results in no between-experts disagreement) and when the client is not very ill it is subjective (in the sense that it results in substantial between-expert disagreement).

Second, very few (if any) people have no medical conditions at all. Almost everyone has something 'wrong' with them, whether it is a sore toe, slightly high blood pressure, or a sub-optimal diet. Why then would an expert doctor ever pronounce someone as being healthy? I suggest that the answer is that a doctor pronounces an examinee healthy if, having diligently searched for medical problems, they cannot find any sufficiently serious illnesses to merit the label “ill”. Whether a problem is sufficiently serious will depend on several factors, and will no doubt vary between contexts (if an opera singer had a sore throat, then the perceived seriousness of the condition may differ to if a bus driver displayed the same symptoms).

Third, a doctor can never be absolutely certain that a person is healthy. There is always the possibility, however remote, that the person that they have examined has some subtle but serious condition that eluded detection. Consequently, if asked to pronounce a potential client "ill" or "healthy", we would expect doctors to be more confident when diagnosing illness (as they would have found a specific condition which they judged serious enough to merit the label "ill") than when diagnosing health (as in this case they would merely have failed to find such a condition).

The illness/health dichotomy is not the only domain in which a negative characterisation is the norm. Consider justice. Although we might not agree what justice is or how to recognise it, there are clear-cut examples of injustices which everybody agrees are unjust, and it is these that concern the legal system.

How do these observations help with the question of proof? The issue I would like to explore is whether the validity of a proof might benefit from being defined negatively rather than positively, as previous researchers have attempted to do.

The negative view of proof

Consider this possible definition:

> (*) A mathematical argument which purports to establish a result is said to be a mathematical proof if no one who has considered the argument has found a serious problem with it yet.

(*) relies upon several assumptions.

First, all mathematical arguments have potential problems. These problems may be overt: the argument may involve the application of an invalid inference (affirmation of the consequent, say), or rely upon a false lemma. The problems may be subtle: there could be an unexplained gap between two parts of the proof which the reader is unsure about how to bridge for themselves. As discussed earlier, problems of the second type are inevitable for real proofs (i.e. non-derivations). Critically, not all of these potential problems need necessarily be sufficiently serious to render the proof invalid.

Second, what is perceived by a mathematician to be a serious problem will depend on many factors: mathematical context, historical context, social context etc. For example, after the so-called "crisis in intuition" period at the turn of the twentieth century, many problems with mathematical arguments which had previously not been considered serious were revisited (Hahn, 1933/1960). Similarly, social context may be important along with historical context: a gap left in a research-level proof may be considered a trivial omission (indeed a desirable omission, if it improves the readability of the text), but in an undergraduate's examination script, the very same gap may be considered serious enough to render the proof invalid (cf. Weber, 2008). Earlier researchers have suggested that there can be disagreements between mathematicians about whether or not purported proofs are valid (e.g., Davis & Hersh, 1983). The negative view suggests that disagreements along these lines are in actual fact disagreements about the existence and seriousness of problems contained within the purported proof.

Several consequences flow from (*):

First, when a mathematician determines that a proof is valid, they can be more or less certain about the lack of serious problems. But they can never be *absolutely* certain that there are no serious problems. Like the doctor pronouncing that a person is healthy, there is always the possibility, with non-zero probability, that a serious problem has been overlooked.

Second, when a mathematician determines that a proof is invalid, they can be certain that there is a problem that they deem to be serious. In other words, the validity of a proof can be objective (in the sense that all competent readers will agree with each other) for invalid proofs, but subjective for valid proofs (in the sense that a proof is only ever conditionally valid: if someone finds a problem with it, at some point in the future it may become invalid). This observation fits with the historical finding that published "proofs" are sometimes later found to be invalid (Davis & Hersh, 1983; de Millo et al., 1979), but that arguments widely perceived to be invalid are rarely later discovered to actually be valid.

Empirically testing the negative view

Can the negative view be empirically tested? I believe it can. It suggests that, like doctors pronouncing on illness/health, a mathematician who decides an argument is invalid should be more confident in their judgement than a mathematician who decides that an argument is valid. In the latter case the mathematician will either have found no problems, or will have found some number of problems which they deemed not serious enough to render the proof invalid. In both of these situations the reader will be worried that they have overlooked some subtle serious problem with the proof. In contrast, when a mathematician deems a proof invalid they will have found a specific problem which they have deemed to be sufficiently serious. We would therefore expect a relationship between the confidence with which mathematicians rate proofs, and their judgements.

In fact, this pattern of results is exactly what Inglis, Mejia-Ramos, Weber and Alcock (2013) found when they tested this prediction. They asked 109 mathematicians to read and evaluate (select "valid" or "invalid") a proof from elementary calculus. Participants were also asked to report the level of confidence with which they made their judgements on a five-point (1 to 5) Likert scale. Two interesting results emerged. First, there was not widespread agreement between the research mathematicians about the validity of the short elementary calculus proof: around a quarter deemed it valid and three-quarters invalid. Second, those who rated it invalid had a mean confidence rating of 4.16/5 (where 5 represents high confidence) compared to 3.41/5 for those who rated it valid (a difference significant at the .001 level). This result makes sense when interpreted from a negative perspective: those who rated the argument invalid had found an explicit problem, their only uncertainty related to how serious it was; in contrast, those who rated the argument valid had an additional source of uncertainty: whether or not they had overlooked a serious problem.

Here I have offered only a sketch outline of how taking a negative view might be useful for thinking about mathematical proofs. A great deal of work would be required to flesh this out so that it became pedagogically or philosophically useful. Considering how a negative view would fit with other non-traditional accounts of proof, notably Dutilh Novaes's (2018) dialogical account, would also be valuable. Nevertheless, the observation that there are empirical findings that are consistent with the negative view's predictions suggests that conducting this work could be worthwhile.

References

Azzouni, J. (2004), The derivation indicator view of mathematical practice. *Philosophia Mathematica*, *12*, 81-106.

Balacheff, N. (1987). Processus de preuves et situations de validation. *Educational Studies in Mathematics, 18*, 147-176.

Davis, P., & Hersh, R. (1983). *The mathematical experience*. Harmondsworth: Penguin.

Davis, P. (1972). Fidelity in mathematical discourse: Is one and one really two? *American Mathematical Monthly, 79*, 252-263.

de Millo, R. A., Lipton, R. J., & Perlis, A. J. (1979). Social processes and proofs of theorems and programs. *Communications of the ACM, 22*, 271-280.

Dutilh Novaes, C. (2018). A Dialogical Conception of Explanation in Mathematical Proofs. In P. Ernest (Ed.), *The Philosophy of Mathematics Education Today* (pp. 81-98). Springer, New York.

du Sautoy, M. (2004). *The Music of the Primes: Why an unsolved problem in mathematics matters*. London: Harper Perennial.

Freiling, C. (1986). Axioms of symmetry: Throwing darts at the real number line. *Journal of Symbolic Logic*, *51*, 190-200.

Hahn, H. (1933/1960). The crisis in intuition. In J. R. Newman (Ed.), *The world of mathematics* (Vol. 3, pp. 1956-1976). London: Allen and Unwin.

Hanna, G. (1991). Mathematical proof. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 54-61). Dordrecht: Kluwer.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics III* (pp. 234-282). Providence, RI: American Mathematical Society.

Inglis, M., Mejia-Ramos, J. P., Weber, K., & Alcock, L. (in press). On mathematicians' different standards when evaluating elementary proofs. *Topics in Cognitive Science.*

Mariotti, M. A., Bartolini Bussi, M., Boero, P., Ferri, F., & Garuiti, R. (1997). Approaching geometry theorems in context: From history and epistemology to cognition. In E. Pekhonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 180-195). Lahti, Finland.

Pelc, A. (2008). Why do we believe theorems? *Philosophia Mathematica*, *17,* 84-94.

Postman, N. (1992). *Conscientious objections: Stirring up trouble about language, technology and education*. New York: Vintage.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*, 4-36.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, *39*, 431-459.