Long-term sense making in arithmetic and algebra[1]

David Tall[2]

In this WikiLetter, I challenge widely held beliefs about the teaching and learning of mathematics by focusing on how humans develop mathematical thinking over the long-term. My purpose is to present an essentially simple approach to teaching and learning which relates to how we make sense of mathematics, in terms of what we perceive through our senses and manipulate within our mind and how we communicate with others.

This requires us to consider how mathematics changes in meaning as it becomes more sophisticated and to focus on those ideas that are 'supportive' — because they continue to work appropriately in a new situation — in contrast to those that are 'problematic' in a new situation because they no longer work in the same way. In particular, we need to focus on supportive ideas that hold good over a longer span of time. These can then operate as a firm foundation in contrast to other ideas that become problematic and can be addressed explicitly rather than become subconscious misconceptions.

In arithmetic, the child learns to count and steadily develops the supportive idea that if a collection of objects is counted, then the number of objects remains the same. However, what matters in the long-term is not just that this works in the case of counting whole numbers, but that the process of addition also has the same supportive property in more sophisticated number systems, from whole numbers through fractions, signed numbers, real numbers and even complex numbers. This may be formulated as:

> **The addition principle**: When adding together a collection of numbers, it does not matter which order you add them, the final result is always the same.

For example, the sum of $2+\frac{3}{4}+1.35+(-17)$ is independent of the order of the terms.

Expressions are spoken *in time* and so have a unique sequence when heard. Meanwhile, written text is read in a conventional sequence which is from left to right in Western societies. If addition and subtraction are performed in a sequence, then the order does not matter. From an expert viewpoint, adding a negative number is the same as subtracting the corresponding positive number. The addition principle at an advanced level therefore extends the simple idea for the young child that a sequence of additions and subtractions does not depend on the order of operations.

A similar principle holds for multiplication:

> **The multiplication principle**: When multiplying together a collection of numbers, it does not matter which order you multiply them, the final result is always the same.

Both these principles hold throughout school mathematics although they fail in more sophisticated contexts such as the product of matrices or of quaternions. On the other hand, when several operations are involved such as addition and multiplication, taking powers, square roots, forming algebraic expressions and so on, the situation becomes more complicated and requires more careful analysis. Let us begin with a simple example:

What is $2+2\times 2$?

If we read this from left to right and perform the operations as they arise in sequence, we start with $2+2$ to get 4 and then calculate 4×2 to get 8. However, in mathematics we use the convention that 'multiplication takes precedence over addition', so we first perform the multiplication 2×2 to get $2+2\times 2=2+4$, giving the 'correct' answer, 6. This rule of precedence is problematic because it violates the natural order of reading and speaking and can cause serious conceptual difficulty. Later, when several operations are involved, the rule is extended to more complicated rules given by mnemonics such as 'BODMAS' (following the order of precedence 'Brackets, 'Of', 'Division', 'Multiplication', 'Addition', 'Subtraction') with variants such as 'BIDMAS' (where the I stands for

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[1] Dedicated to my now 12-year-old grandson, Simon Tall.

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'Index') or in the USA by 'PEMDAS' (`Parentheses', 'Exponents', 'Multiplication', 'Division', 'Addition', 'Subtraction') remembered using the phrase 'Please Excuse My Dear Aunt Sally'. When these rules are learnt by rote and used procedurally, they may work for a few individuals, but they cause significant difficulties for most learners as the mathematics becomes more sophisticated.

The 'New Mathematics' of the 1960s attempted to clarify matters by defining specific laws of arithmetic such as the 'commutative', 'associative' and 'distributive' laws which apply to 'binary' operations of addition and multiplication. The laws were then 'generalised' to apply to more general expressions in arithmetic and algebra as new concepts are introduced involving place value, fractions, signed numbers, decimals, real numbers (as infinite decimals), which may be visualised as points on a number line and even to complex numbers, as points in the plane. Mathematics Education developed into an industry specifying difficulties encountered as 'misconceptions' to be corrected. I would like to offer a different idea for dealing with matters and conceptualizing difficulties.

Making sense of arithmetic by how we say symbolic expressions

It turns out that we can make sense of these procedural rules simply by paying attention to the way in which we *speak* them and how others *hear* what we say. For example, there are (at least) two different ways in which we can speak the expression '$2 + 2 \times 2$' by leaving tiny gaps between various words.

In order to represent how phrases can be articulated, I will write two dots $(..)$ between symbols, so that '$2 + 2..\times 2$' is spoken as 'two plus two [gap] times two' which can be interpreted as $2 + 2$ (which is 4) times 2, giving the final answer 4×2 which is 8.

By the same token, $2..+2 \times 2$ (or $2+..2 \times 2$) means $2 + 4$, which is 6.

You should speak the two expressions '$2 + 2..\times 2$' and '$2..+2 \times 2$' out loud to yourself and to someone else, to see how these two ways of speaking give two clearly different meanings both to yourself and to someone else. *Do this now before proceeding*.

This leads to a further principle which applies not only in arithmetic but also in algebra: different meanings can be expressed by the manner in which an expression is *spoken* and *heard*, giving:

> **The articulation principle**: The meaning of a sequence of operations can be expressed by the manner in which the sequence is *articulated*.

This principle has wide ranging applications throughout mathematics. I only realised its power when my then 11-year-old grandson, who already knew about powers of negative numbers, teased me by asking, 'What is the square root of 9 times 9?', spoken evenly without any implied articulation. I knew he was familiar with squares of negative numbers, so I replied that the answer could be $+9$ or -9.' 'No,' he replied, 'it is 27.' I was shocked by his unexpected answer. Then he explained that he meant 'the square root of 9 .. times 9,' articulating the expression as 'the square root of 9' (which is 3) times 9, which gives his final answer as 27.

At a stroke, this young child had opened up the door to a whole new way of making sense of the operations in arithmetic and algebra. Instead of learning arbitrary rules of precedence, he offered a new way of making meaningful sense by focusing on how to *speak* mathematical expressions and to communicate the ideas in ways that other people could *hear*.

Problematic aspects of operations other than the principles of addition and multiplication occur throughout mathematics. For example, subtraction can be articulated in ways that have different meanings. The subtraction $5 - 3 - 1$ has different answers. When it is spoken as '$5 -..3 - 1$', it can be interpreted as $5 - 2$, and written with brackets as $5 - (3 - 2)$, giving $5 - 1$ which is 4. When spoken as $5 - 3..-1$ it is $2 - 1$, which is 1.

General applications of the articulation principle

Over the weeks and months that followed the insight given me by my young grandson, I found that the principle applies widely throughout mathematics involving combinations of operations. For instance, in McGowen and Tall (2013), we studied the difficulties that students had in dealing with quadratic expressions which included various misconceptions, such as when a quadratic expression was evaluated for a negative number, say calculating x^2 when $x = -2$, then some students interpret this as 'the square of minus 2' while others would see it as 'minus the square of 2'. More generally the expression $-x^2$ might be interpreted as 'minus .. x squared' which is $-(x^2)$

or

'$minus\ x$ .. squared' which is $(-x)^2$.

This offers a completely new way of interpreting the data. Instead of seeing student difficulties in terms of 'misconceptions' that arise in terms of the conventional use of brackets, they could be seen in terms of the different ways of speaking (and hearing) the symbols.

Of course, there are still conventions to be introduced to minimise the length of expressions, such as writing the product of 2 times x as $2x$ (omitting the multiplication sign where this need not cause confusion). There are even more subtle uses of symbols where $2x$ means 2 $times\ x$, 2½ means 2 $plus$ ½ and 21 means 2 $times$ 10 $plus$ 1. Even so, using the addition and multiplication principles allows us to simplify how we read expressions in arithmetic and algebra.

Binding

Reading more complicated expressions such as

$$2x^2 + 7\mathrm{x} + 5 + 3x^2 - 3\mathrm{x}$$

presents a new level of sophistication. By the addition principle, seeing the expression as a sequence of addition and subtraction of terms, the order does not matter. It could be rewritten as $2x^2 + 3x^2 + 7\mathrm{x} - 3\mathrm{x} + 5$. If we imagine the term x^2 as a single entity, then we can add two of them ($2x^2$) to three of them ($3x^2$) to get 5 of them ($5x^2$). Similarly, we can combine the terms 7x − 3x to get 4x.

Notice that, as we perform these operations, we see a term such as $3x^2$ written not just linearly as successive symbols, but that the power ^{2} is raised up as a superscript.

In more sophisticated examples such as

$$\sqrt[3]{\frac{4x^3-2}{x^2+1}} \qquad (1)$$

the symbols are placed in a spatial array rather than a linear sequence. In this case, it may be written linearly, and spoken sequentially, as 'the cube root of ((four x cubed minus two) over (x squared plus one)), where the brackets denote how the symbols are grouped together. At this point it is easier to write and read rather than to say or hear. At this point, the visual embodied sense, which is an essential part of the human conceptualisation, blends with the symbolic sense.

To be able to read such a symbol requires it to be scanned and interpreted. For example, in expression (1), the powers $x^3$and x^2 are each strongly bound together as single entities. The numerator $4x^3 - 2$ is the difference between two terms $4x^3$ and 2, while the denominator $x^2 + 1$ is the sum of two terms x^2 and 1.

Making sense of such expressions requires far more sophistication than the procedural use of a mnemonic such as BODMAS or PEMDAS. Climbing the conceptual staircase from the child's initial experience of sorting and counting to successively more sophisticated expressions in arithmetic and algebra requires a succession of subtle changes in meaning. Initially this can build on the addition and multiplication principles coupled with insights from the articulation principle. This can then be translated into the meaningful use of brackets to indicate subtle meanings for linear sequences of terms. This is followed by introducing conventions to shorten expressions using implicit multiplication and spatial layout to represent more complicated expressions.

Some conventions are less obvious. For example, the expression $e^{x^2}$is usually read as 'e to the power x^2' rather than 'e^x squared'. (To convince yourself of this, consider what the expression e^{x^2+x} means to you.)

If a power is written using the notation x^n for x^n, then the power e^{x^2}may be written as e^x^2. However, if brackets are introduced, then e^(x^2) is different from (e^x)^2. Writing the expression as e^(x^2) is called 'right binding' because the two terms on the right are bound together and this operation takes precedence. The form (e^x)^2 is called 'left binding'. The usual interpretation of 2^{3^4} is 2^(3^4) = 2^81 which uses right binding and gives a different answer from reading in the usual left to right direction where (2^3)^4 = 2^12. This is very different from addition and multiplication, where left binding and right binding give the same result: $(2 + 3) + 4$ is the same as $2 + (3 + 4)$ and $(2 \times 3) \times 4$ is the same as $2 \times (3 \times 4)$.

Summary

What does this tell us about sense making in arithmetic and algebra over the long term? It tells us that sense making may initially be clarified by paying attention to the verbal articulation of expressions involving addition and multiplication. Over the longer term **the principles of addition and multiplication** show that the final sum or the final product of a collection of numbers remains the same whatever the order. These principles remain stable and supportive throughout school mathematics whatever kinds of number are involved, be they whole numbers, fractions, signed numbers, decimals, real or complex numbers. On the other hand, when other operations are involved, different articulations, which correspond to different meanings, and hence, to different ways of grouping terms in brackets, invariably give different results as formulated by **the articulation principle**. These meanings can be written using brackets to express the different articulations so that problematic aspects of other operations and combinations of operations can be made explicit. Instead of introducing rote-learned conventions that violate the natural sequence of spoken language, it becomes possible to build on meaningful human experience and prepare a longer-term path for meaningful learning for a wider proportion of the population.

This approach has not yet been the subject of any clearly focused research project that I know. You, dear reader, will have your own interpretation of what I have suggested here based on your own experience, as I am also biased by my own experiences as a pure mathematician, as a participant in teacher training, and as a parent and grandparent observing the long-term growth of my own children and grandchildren. It is time for us all to rise above our own specific experiences in our own familiar surroundings and to seek a broader vision of what it means to learn to think mathematically in an ever changing world.

In this short writing, I focused on the essential blending of linear symbolism and visual embodiment in the long-term sense making in arithmetic and algebra. For further reading, the full framework of long-term learning in mathematics as I understood it in 2013 is given in my book *How Humans Learn to Think Mathematically* (Tall, 2013); the story of the origin of my interest in articulation is given in Tall, Tall & Tall (2017) and a more recent exposition relating to long-term arithmetic and algebra is given in a paper written for Japanese elementary school-teachers (Tall, 2018).

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