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It's gonna reach out and grab ya
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Algebradabra

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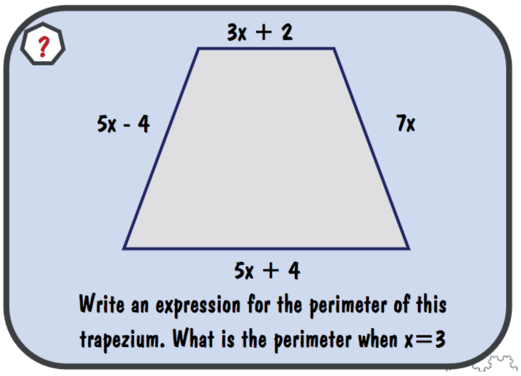
In this short essay I want to write about an algebra blog that I recently started (https://algrabya.blogspot.co.uk/). On it I post, and comment on, a weekly set of five algebra tasks. I also release each day's version of the tasks on Twitter (using the handle @ProfSmudge, my pen name for the *Maths Medicine* pocketbooks). The tasks are broadly aimed at lower secondary school students, though I hope they will also engage other age groups as well as their teachers. My motivation for doing this is twofold. I am semi-retired so need something to do! More importantly, I feel it is hard to find 'good' school algebra tasks, be it in textbooks or on Twitter, so I have tried to write some. I should add that by 'good' I am thinking here principally of tasks that help students develop a feel for algebra rather than tasks that focus on procedures, although helping students make sense of algebra should also help in the learning of procedures. In terms of Kieran's (2007) school algebra typology of *generational*, *transformational* and *global/meta level*, my focus therefore is mainly on the generational.

In designing tasks for the blog, I have built particularly on my work since 2008 on the ICCAMS design-research project, where I helped develop teaching materials aimed at lower-secondary school students. [ICCAMS stands for *Increasing competence and confidence in algebra and multiplicative structures* - see http://iccams-maths.org/] However, I have tried to avoid producing tasks that are too similar to the ICCAMS tasks.

School algebra is often presented in a fragmented way, with the emphasis on the manipulation of algebraic symbols and on performing procedures (for solving equations, for finding the gradient of a line, etc). Gaining fluency in such activities can be empowering, but I think the opposite often occurs: I believe we can seriously disable students when such activities seem to have little meaning or purpose.

Consider the task below. Tasks like this are widespread on Twitter and in textbooks, but in my view they embody many of the things that are often deficient in school algebra tasks. The intention behind the task is to give students the opportunity to practise algebraic manipulation. The occasional use of tasks for this purpose is OK (perhaps), and no doubt students understand that this is what the task is for. However, the task will surely also convey other messages to students. For a start, that algebra lacks 'purpose and utility' (Ainley & Pratt, 2002). How has it come about that the lengths of this trapezium's sides are



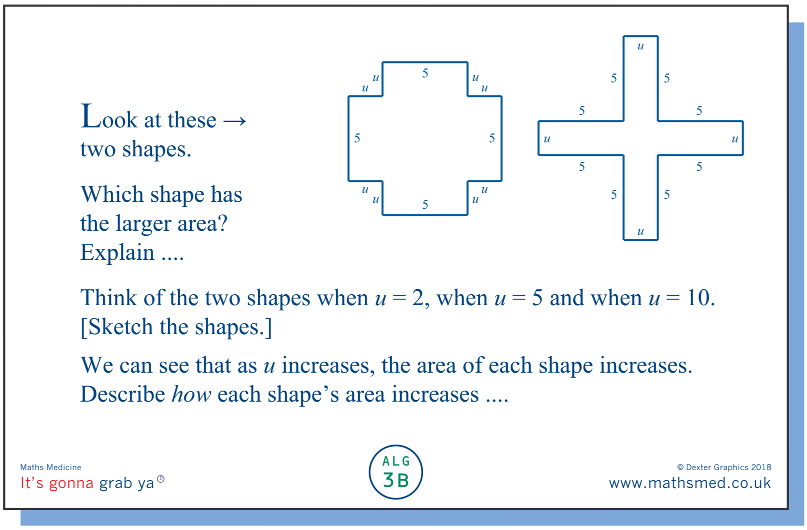
given by these diverse expressions? There is no obvious sense to this, so it is difficult to see a purpose in finding the shape's perimeter, be it a practical purpose of solving a meaningful 'real-life' problem or an intriguing purpose of exploring an engaging piece of mathematics. What makes matters worse is that there is no genuine interest in the geometric context in which the task happens to be set. It is merely there to dress up the fact that we want students to add the four given expressions. Does that make the task any more engaging, or connected, or meaningful? I don't think so. And what does it convey about the value we place on geometry?

It should be said, the resulting expression for the perimeter ($20x + 2$) is quite neat so the task could provide a hint of the utility of algebra for simplifying information - had there been any purpose in doing so.

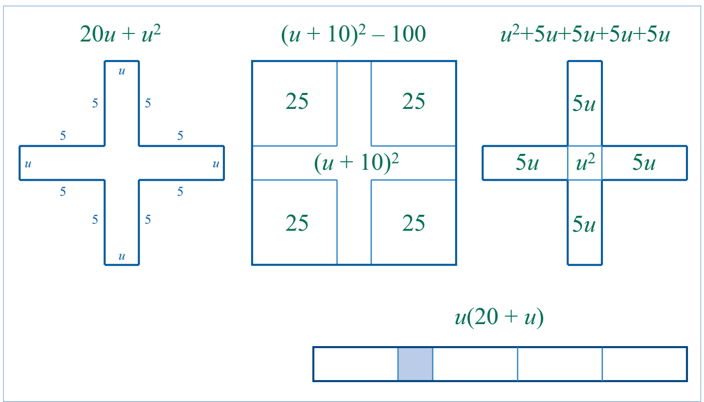
The task's saving grace, perhaps, is the second part, "What is the perimeter when $x = 3$?". For some students at least, this may convey the idea that x can take on several values, ie that we can use a letter to represent a *variable*, not merely a specific number that happens, initially, to be unknown - as is so often the case in school algebra.

We can rescue the above task if we respect the geometry. What actually happens to the shape when $x = 3$? Is it still symmetrical, as suggested by the diagram? Is a trapezium with parallel sides of 11 and 19 units, and slanting sides of 11 and 21 units, even possible? It turns out that it is not! This underlines the disregard for sense-making exhibited by the designers of the task. However, we could still make something positive out of this by investigating the range of values of x for which the shape does exist. And we could look at this more deeply by, for example, examining what happens to the length of each side, and to the shape as a whole, as x changes by a given amount.

In one of our set of 5 weekly blog tasks we compare the areas of two cross-shapes, some of whose edges can vary. The figure below shows the second of the five tasks.



It can be said, just as with the earlier perimeter task, that we provide no rational for the existence of these strange shapes. How have they arisen? Why do they exist? It might be possible to invent a practical rational, but I would argue that the task is valid because it is intriguing. Visualising the shapes as the length u of some of the edges varies is challenging (although interestingly, it becomes a lot easier if one sees the shapes as the nets of open boxes, an idea we introduce in version 3). Comparing the shapes' areas is also challenging: it turns out that as u varies, the areas vary at different rates and the relative size of their areas flips at a particular value of u. For one of the shapes the relation is linear, for the other it is quadratic. One can get a sense of this from the fact that, when viewed as open boxes, one box simply gets taller while for the other its square 'base' expands, ie gets larger in two directions. In version 4 we present the expression $20u + 25$ for the area of one of the shapes (Which one?) and ask students to find an expression for the area of the other shape. A nice thing about this activity is that it can be done by thinking about the geometry of each shape in a variety of ways (see below). This results in different but equivalent expressions which in turn provides an incentive to manipulate the expressions to show that they are indeed equivalent.



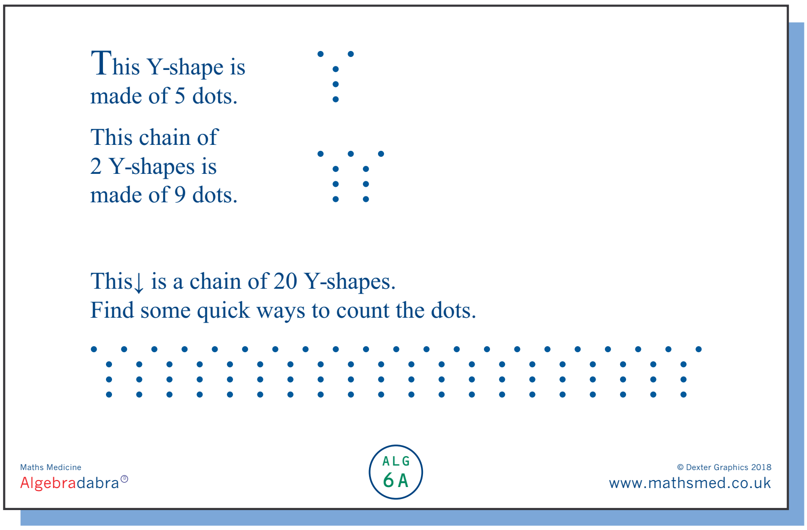
The 'dynamic' nature of the cross-shapes set of tasks is deliberate, and is an approach that we frequently use in ICCAMS to help students develop a feel for the notion of 'variable'. However, such an approach is often neglected in school algebra. And, as mentioned earlier, much of school algebra is concerned with working with specific (but temporarily unknown) numbers rather than with variables. So it is a moot point to what extend such work can be said to concern generality and structure.

An area that does involve generality is that of functions and their (Cartesian) graphs. However, this is often approached in a fragmented way - we often seem unconcerned with what a function is 'about' or how it 'behaves'; instead, when its graph is a straight line (as it usually is!) we tend to focus on the procedural rule that states that the line's equation can be written in the form $y = mx + c$, where m is the line's gradient ('rise over run') and c the place where the line cuts the y-axis. One of the most effective ICCAMS lessons is *Boat hire*, which involves a story about two rules for hiring a boat: '£5 per hour' or '£1 per hour plus a flat fee of £10'. "Which would you choose?" We have found that the task intrigues students and that the story is easily accessible to them even if they have never hired a boat and perhaps will never do so. The lesson integrates the story with the use of tables of values, algebraic expressions and graphs, each of which is used to illuminate the others.

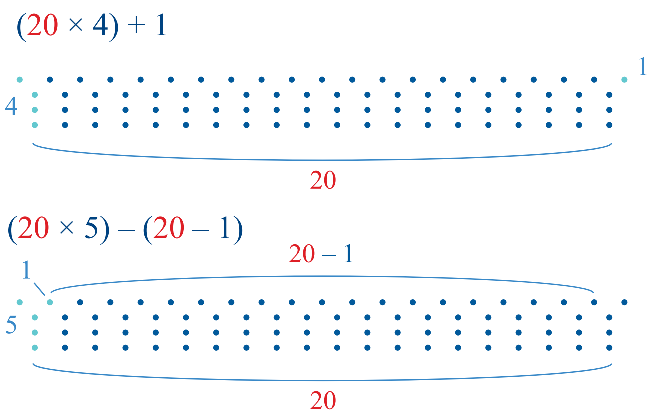
I have not used a task like *Boat hire* in my blog, not wishing to duplicate the ICCAMS materials. However, I have tackled the issue of generality in other contexts, in particular through the use of figurative patterns. Examples of figurative patterns can be found in many textbooks, though they often play quite a minor role when one considers how central generalisation should be to school algebra. Where they do occur, the patterns are commonly presented as *growth patterns*: typically, we are shown a sequence of three or four members of a pattern-type (called the 'first, second and third' members) and we are then asked to draw the next pattern-member and then to find the number of certain elements within the pattern in, say, the 20th, the 100th and the nth member. This approach allows for a certain amount of exploration which can be helpful when looking for the structure of a pattern. However, it can also easily lead to an empirical approach, where ordered pairs of numbers are listed in a table and a rule is sought ('within' or 'between' the numbers) without reference to the actual structure of the pattern.

To help keep the focus on structure, I am keen on using patterns *generically* when appropriate (Küchemann, 2010). In the task below, a 'Y-shape' is defined and we are shown how it is used to determine a pattern-member's number. We then immediately jump to a fairly 'distant' member - the 20th. By asking students to find efficient ways of counting the number of dots in this member we are in effect asking students to *focus on this one member* and to *structure* it.

The strength of this kind of task is that the pattern can be structured in seemingly different ways which turn out to be equivalent. As with the cross-shapes, this provides an incentive to symbolise the structurings and to manipulate them in order to show their equivalence. In the second version of the task we offer these two expressions for the



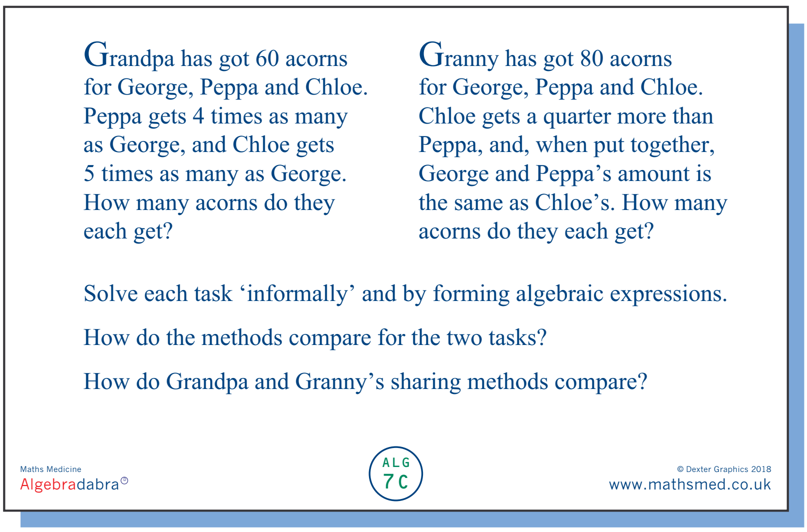
structure of the 20th member and then ask for a third expression: (20 × 4) + 1 and (20 × 5) – (20 – 1). Note here that we are using 'open sentences', with '20' acting as a quasi variable. These expressions, of course, arise from the spatial arrangement of the dots, as illustrated below.



So far I have produced 16 sets of tasks for the blog. I don't know whether I will be able to come up with many more - though it would be nice to reach 20, making 5×20 variants in all!

In the meantime, I will close this essay with one more task, shown below. Though it involves situations that are highly contrived, it offers one way of showing the utility of algebra.

The story on the left is in some ways reminiscent of the perimeter task with which I began this essay. If we decide to solve the task



using algebraic symbolisation (eg, by letting g stand for the number of George's acorns), we get an equation involving the sum of several terms which we can then simplify: $60 = g + 4g + 5g = 10g$. (So $g = 6$ and George, Peppa and Chloe get 6, 24 and 30 acorns respectively.) Of course, the story is quite artificial, silly even, but it has a valid purpose: it's a puzzle. However, we clearly don't need to use formal algebra here and that is a dilemma with many school algebra tasks - they can often be solved informally, so why struggle with trying to learn formal methods?!

The second story can also be solved informally, but less easily. It involves a 'circular reference', a problem-structure that I first came across in the excellent review of school algebra by Mason and Sutherland (2010), and which can be traced back to Euler's *Elements of Algebra* (or possibly to additions in a French translation by de la Grange).

We can symbolise the second story in this kind of way: $g + p + c = 80$, $c = p + p/4$, $g + p = c$. This looks complicated, but at least we've got the various relations down on paper - we don't need to keep track of them in our head anymore. All that remains is to manipulate the symbols (if we have the skills to do so!) until the desired values emerge .... Even though the letters here represent specific unknowns rather than variables, this method of solution might help students see how powerful the use of algebraic symbolisation (and procedures) can be.

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