A KEY TO DEVELOPING ALGEBRAIC THINKING: SEEKING, USING, AND EXPRESSING STRUCTURE

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1 Introductory Remarks

High school algebra involves working with generalized forms. The ability to see structure in these forms is crucial to being successful in algebraic transformational activity and to making sense of those transformations. While generalization has long been considered the heart of school algebra, this focus on the process of generalizing has to a large extent obscured the process of seeing structure. Attention needs to be paid to the complementary process of *looking through mathematical objects*, such as the expression $x^6 - 1$ or the number 989, and to decomposing and recomposing them in various structural ways (e.g., seeing that $x^6 - 1$ can be decomposed into $(x^3)^2 - 1$ or into $(x^2)^3 - 1$ and factored accordingly, or seeing that 989 can be decomposed into, for example, the structural expressions $9 \times 109 + 8$ or $9 \times 110 - 1$, or even $9 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$). As the latter example suggests, attention to looking through mathematical objects can mean developing awareness of the possible and various ways of structuring number and the numerical operations of arithmetic – arithmetic being a precursor to high school algebra. However, as Arcavi and his collaborators (2017) have suggested, students' experiences in learning arithmetic only rarely foster an appreciation of structure. Similarly, Mason (2016) has pointed out that looking at something structurally is an often-overlooked aspect of algebraic thinking (algebraic thinking being a component of both arithmetic and algebra). This paper argues for the importance of a structural perspective in the development of algebraic thinking by first exploring what is meant by algebraic thinking, then examining the notion of structure from various theoretical and mathematical perspectives, and finally offering research-based commentary on drawing out structure with the aim of developing algebraic thinking.

2 Algebraic Thinking Within Both Arithmetic and Algebra

The learning and teaching of school algebra has traditionally involved the secondary school student (approximately 12 to 18 years of age) and has focused on forming and operating on polynomial and rational expressions, on representing word problems with algebraic expressions and equations containing variables and unknowns, and on solving algebraic equations by means of axiomatic and equivalence properties. However, over the past several decades, changes in perspective as to what constitutes school algebra have occurred, with the result that several different conceptualizations of school algebra

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have emerged. For example, Arcavi et al. (2017) define the aims of school algebra as including “expressing generalizations, establishing relationships, solving problems, exploring properties, proving theorems, and calculating” (pp. 2-3). In Stacey and Chick (2004), school algebra is seen as “a way of expressing generality; a study of symbol manipulation and equation solving; a study of functions; a way to solve certain classes of problems; and a way to model real situations” (p. 16). The lack of universality regarding definitions of school algebra is emphasized by Leung et al. (2014) who provide evidence that algebra lessons around the world can vary not only from country to country, but also within country, and that this diversity can be characterized not just in terms of content but additionally as to whether the main focus is either procedural or conceptual or some combination of the two.

Some years ago, Freudenthal (1977) characterized school algebra as consisting of not only the solving of linear and quadratic equations but also algebraic thinking, which he stated includes the ability to describe relations and solving procedures in a general way. This latter facet highlighting algebraic thinking, quite novel at the time, opened up additional dimensions for conceptualizing school algebra at the secondary level so as to include consideration of algebraic thinking processes and also provided an avenue for developing an algebraic thread in elementary school mathematics. As noted in Kieran et al. (2016), the interest in fostering algebraic thinking at the elementary school level has steadily evolved over the past 20 years or so to include a focus on mathematical relations, patterns, and arithmetical structures, with detailed attention to the reasoning processes used by young students, aged from about 5 to 12 years, as they come to construct these relations, patterns, and structures – processes such as noticing, structuring, conjecturing, generalizing, representing, and justifying. A notable aspect of this activity with the younger student is the use of alternatives to alphanumeric symbols (e.g., words, artefacts, or other mathematical signs) for the expression of generality involving indeterminate objects (see Radford 2018). To sum up, contemporary notions of algebraic thinking within both arithmetic and algebra, while remarkable for their diversity, embrace on the one hand, sign-based activity involving mathematical objects and the relations among them and, on the other hand, the mathematical thinking processes underpinning such activity. However, within these contemporary notions, the aspect of seeking and expressing structure has not been accorded the same privileged position as has the process of generalizing.

3 On Structure

Structure is without doubt one of the big ideas of mathematics and is to be found everywhere in mathematics. While it might be assumed that there is widespread agreement on the meaning of the term *structure*, nuances abound. Some researchers view the seeing of structure as preliminary to the act of generalizing. For example, Blanton and her colleagues (2011) state that, "generalizing is the process by which we identify structure and relationships in mathematical situations. … It can refer to identifying relationships between quantities that vary in relation to each other. It can also mean lifting out and expressing arithmetic structure in operations on the basis of repeated, regular observations of how these operations behave" (p. 9).

But, conversely, structural activity involves identifying that which is general, according to Mason and his coauthors (2009):

> We take *mathematical structure* to mean the identification of general properties which are instantiated in particular situations as relationships between elements; these elements can be mathematical objects like numbers and triangles, sets with functions between them, relations on sets, even relations between relations in an ongoing hierarchy. Usually it is helpful to think of structure in terms of an agreed list of properties, which are taken as axioms and from which other properties can be deduced. … When a relationship is seen as instantiation of a property, the relation becomes (part of) a structure. (p. 10)

For Mason et al. (2009), attending to properties lies at the core of structural thinking, the latter of which they define as a disposition to use, explicate, and connect properties in one's mathematical thinking. If a relationship between two or more objects is not seen as exemplifying some general property, then that relationship is not in itself related to structural thinking. They assert that "structural appreciation lies in the sense of generality, which in turn is based on basic properties of arithmetic such as commutativity, associativity, distributivity and the properties of the additive and multiplicative identities 0 and 1, together with the understanding that addition and subtraction are inverses of each other, as are multiplication and division" (p. 15).

However, Freudenthal (1983, 1991) goes beyond the basic properties of arithmetic in his discussions of structure. He emphasizes that the system of whole numbers constitutes an *order structure* where addition can be derived from the order in the structure, such that for each pair of numbers a third, its sum, can be assigned. The relations of this system are of the form $a + b = c$, which he calls an *addition structure*. The *multiplicative structure* of the natural numbers is described in terms that comprise more than the act of multiplying. It is the whole complex of relations $a \times b = c$, possibly also expressed as $c / b = a$, and complemented by $a \times b \times c = d$, $a \times b = d / c$, and all other relations one would like to consider in this context. It encompasses such properties as commutativity, associativity, distributivity, equivalence of $a \times b = c$ and $c / b = a$, and many more properties of this kind. But, according to Freudenthal, the structure of the natural numbers also allows for prescribing c in the relation $a \times b = c$ and asking for its splittings into two factors. Freudenthal asserts further that c can be split into its prime factors, with divisors and multiples being other means of structuring. As well, tying the order structure to the multiplicative structure yields the property that, given the product, increasing one factor means decreasing the other.

To summarize, in Freudenthal's discussion of structure there is not just one all-encompassing *structure*. There is order structure, additive structure, multiplicative structure, structure according to divisors, structure according to multiples, and so on. And these different but related structures have properties – in fact, *many* properties based on these structures, not simply the basic properties of arithmetic that are often referred to as the field properties. We notice too that Freudenthal also uses the phrasing, *means of structuring*, which puts forward the notion of alternative structurings that can be deduced from the basic structures. Freudenthal's perspective serves to broaden considerably the dimensions of any discussion related to characterizing structures and structuring activity within the mathematics of arithmetic and algebra and where the development of algebraic thinking is a goal.

4 On Structure in Research on Algebraic Thinking

Arguing for the importance of an enlarged perspective on the meaning of structure and its crucial role in the development of algebraic thinking calls for revisiting the body of research on algebra learning to see how it might resonate with, and perhaps even serve to fine-tune, the proposed point of view on seeking and expressing structure. In their investigations of structure, Hoch and Dreyfus (2004) defined *algebraic structure* as follows:

> Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure. (p. 50)

One of the examples they provide is the expression $30x^2 - 28x + 6$ that students come to see as having a quadratic structure, which in turn allows it to be transformed into an equivalent factorized expression involving two linear terms. Their definition alerts us to the aspect of internal order, as well as to its possible structural decompositions. Warren (2003), in a paper on the role of arithmetic structure in the transition from arithmetic to algebra, draws attention as well to the properties related to equivalence and equality. Linchevski and Livneh (1999), who coined the phrase "structure sense," maintain that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. These researchers thereupon specify that instruction be designed to foster the development of structure sense by providing experience with equivalent structures of expressions and with their decomposition and recomposition.

Some current research on the development of algebraic thinking at the elementary school level (see, e.g., contributions in the volume edited by Kieran, 2018) includes a focus on the specific structural aspects of decomposition, recomposition, and substitution within number and numerical operations. In other studies, patterning activity and functional situations serve as means for directing students' attention to these structural aspects. At the secondary level, to add to the structure-inducing practices recommended above by Linchevski, Livneh, and others, more recently Star et al. (2015) have also emphasized the importance of teaching students of algebra to seek, use, and express structure, even if from a somewhat different perspective:

> *Teach students to utilize the structure of algebraic representations*, [by] (i) promoting the use of language that reflects mathematical structure; (ii) encouraging students to use reflective questioning to notice structure as they solve problems; and (iii) teaching students that different algebraic representations can convey different information about an algebra problem. (p. 2)

In brief, the growing attention to the importance of noticing structure within both arithmetic and algebra – in conjunction with the already widespread advocacy of the importance of generalizing – offers support for a more global view related to developing algebraic thinking. This more global view has a dual face: one face looking towards generalizing, and, alternatively but complementarily, the other face looking in the opposite direction towards "seeing through mathematical objects" and drawing out relevant structural decompositions.

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